NECESSARY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS: THEORY AND APPLICATIONS

by

Md. Haider Ali Biswas

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UNIVERSIDADE DO PORTO

Department of Electrical and Computer Engineering

Faculty of Engineering

University of Porto, Portugal

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Dedication

To my

wife ***Nilufa Yesmin***

and

daughters ***Anika*** and ***Tahsin***...

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Abstract

In this thesis, we study both theory and applications of optimal control problems with constraints. First we study the effect of constraints on optimal control problems via the study of a real problem for the control of infectious diseases based on the so called "SEIR" model. We show that the introduction of mixed and state constraints gives rise to new and more realistic vaccination strategies. In the second part of the thesis we derive two new sets of necessary conditions of optimality for optimal control problems with state and mixed constraints in the form of nonsmooth maximum principles. We first consider the problem with state constraints. Our results are then generalized to cover the case where mixed constraints in the form of inequalities are presented.

The new nonsmooth maximum principles for state constraints and then for state and mixed constraints are derived first for convex problems and then extended to the non-convex case. The main ingredient of these necessary conditions is the "joint" adjoint inclusion with respect to both state and control variables. Nonsmooth maximum principle for optimal control problems when both pure state and mixed constraints are present, is sparse in the literature. The novelty of our more general result is that it extends previously obtained results with mixed constraints to cover the case where additional state constraints are present. Of special notice is that our nonsmooth necessary conditions of optimality keep the novelty of being also sufficient when applied to normal linear convex problems.

In context of applications, we consider only smooth problems. This is because our main concern is to illustrate how constraints affect the solution of optimal control problems. We consider what we believe to be realistic modifications of an optimal control problem based on a "SEIR" epidemic model to study vaccination policies to control the spreading of an infectious disease. We solve the problems numerically using well known solvers. The numerical results for mixed constraints case are then partially validated by the theory. In particular normality of the minimizers is verified and the analytical and computed multipliers compared.

Sumário

Esta tese centra-se na teoria e aplicações de problemas de controlo ótimo com restrições. Na primeira parte da tese investigamos os efeitos da introdução de restrições em problemas de controle ótimo num problema real que modeliza o controlo por vacinação de doenças infecciosas usando um modelo "SEIR". Verificamos que a introdução de restrições mistas e de estado dá origem a estratégias de vacinação mais realistas. Na segunda parte da tese derivam-se dois novos conjuntos de condições necessárias de otimalidade para problemas de controle ótimo com restrições de estado e restrições mistas na forma de princípios máximos não suave. Primeiro estes resultados são obtidos para problema com restrições de estado sendo depois generalizados para abranger com restrições mistas e de estado na forma de desigualdades. As novas versões de princípios máximos não suave para problemas com restrições de estado e de estado e mistas são derivadas primeiro assumindo convexidade. Estes são, de seguida, estendidos ao caso não convexo. Uma característica importante das nossas condições necessárias é a inclusão adjunta que é apresentada englobando o estado e o controlo. Note-se que a literatura sobre Princípio do Máximo não suave para problemas de controle ótimo com ambas as restrições mistas e de estado é escassa. A novidade do nosso resultado mais geral é a generalização de resultados anteriores para cobrir o caso em que restrições de estado estão presentes. Mais ainda, as nossas condições necessárias mantêm a característica de serem suficientes quando aplicadas a problemas normais e linear convexos.

No capítulo sobre aplicações, consideramos apenas problemas suaves. Isso porque a nosso principal preocupação é ilustrar como restrições afectam a solução de problemas de controlo ótimo. Consideramos o que pensamos ser modificações realistas de um problema de controle ótimo baseado num modelo "SEIR" para estudar políticas de vacinas para controlar a propagação de uma doença infecciosa. Resolvemos os problemas numericamente usando solvers bem conhecidos. Os nossos resultados numéricos para o caso de restrições mistas são então parcialmente validado pela teoria. Em particular a normalidade dos minimizadores é verificada e os valores dos multiplicadores calculados analiticamente e numericamente são comparados.

Nomenclature

$x+\varepsilon\mathbb{B}$	Ball of radius ε centered at x in Euclidean space
$\mathbb{B}$	Closed unit ball in Euclidean space
$d_C(x)$	Euclidean distance of x to the set C
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{R}$
$\lVert \cdot \rVert_X$	Norm in the space X
$\lVert \cdot \rVert_1$	The norm of $L^1([a,b];\mathbb{R}^p)$
$\lVert \cdot \rVert_\infty$	The norm of $L^\infty([a,b];\mathbb{R}^p)$
$\lVert \mu \rVert_{TV}$	The norm of $C^*([a,b];\mathbb{R})$
NCO	Necessary Conditions of Optimality
OCP	Optimal Control Problem
CQ	Constraint Qualification
MP	Maximum Principle
PMP	Pontryagin Maximum Principle (or Smooth/Classical Maximum Principle)
NMP	Nonsmooth Maximum Principle
Ψ_A	The indicator function of a set A
$N_A(x^*)$	Normal cone to a set A at the point x^*
$N_A^L(x^*)$	Limiting normal cone (also known as Mordukhovich normal cone) to A at x^*
$N_A^C(x^*)$	Clarke normal cone to a set A at the point x^*

$\partial^L f(x^*)$	Limiting subdifferential of a function f at the point x^*
$\partial^C f(x^*)$	Clarke subdifferential of a function f at the point x^*
$\bar{\partial}_x h$	The subdifferential of a function h with respect to state variable x
$\partial_x^> h$	The hybrid subdifferential of a function h with respect to state variable x
$C([a,b];\mathbb{R})$	Space of continuous functions from $[a,b]$ to $\mathbb{R}$
$C^*([a,b];\mathbb{R})$	Dual space of the space of continuous functions $C([a,b];\mathbb{R})$
$C^\oplus([a,b];\mathbb{R})$	Set of nonnegative elements in $C^*([a,b];\mathbb{R})$ on nonnegative valued functions in $C([a,b];\mathbb{R})$
$W^{1,1}([a,b];\mathbb{R})$	Space of absolutely continuous functions from $[a,b]$ to $\mathbb{R}$
$L^1([a,b];\mathbb{R}^p)$	Space of integrable (or L^1) functions from $[a,b]$ to $\mathbb{R}^p$
$L^\infty([a,b];\mathbb{R}^p)$	Space of essentially bounded (or L^∞) functions from $[a,b]$ to $\mathbb{R}^p$
$C^{1,1}$	Class of continuously differentiable functions with locally Lipschitz continuous derivatives
(x^*,u^*)	Optimal solution over all admissible processes for an optimal control problem
Grf	The graph of a function (or multifunction) f
epi$f$	The epigraph of a function (or multifunction) f
dom$f$	The domain of a function (or multifunction) f
bdy$S$	The boundary of a set S
cl$S$	The closure (also denoted by $\overline{S}$) of a set S
int$S$	The interior of a set S

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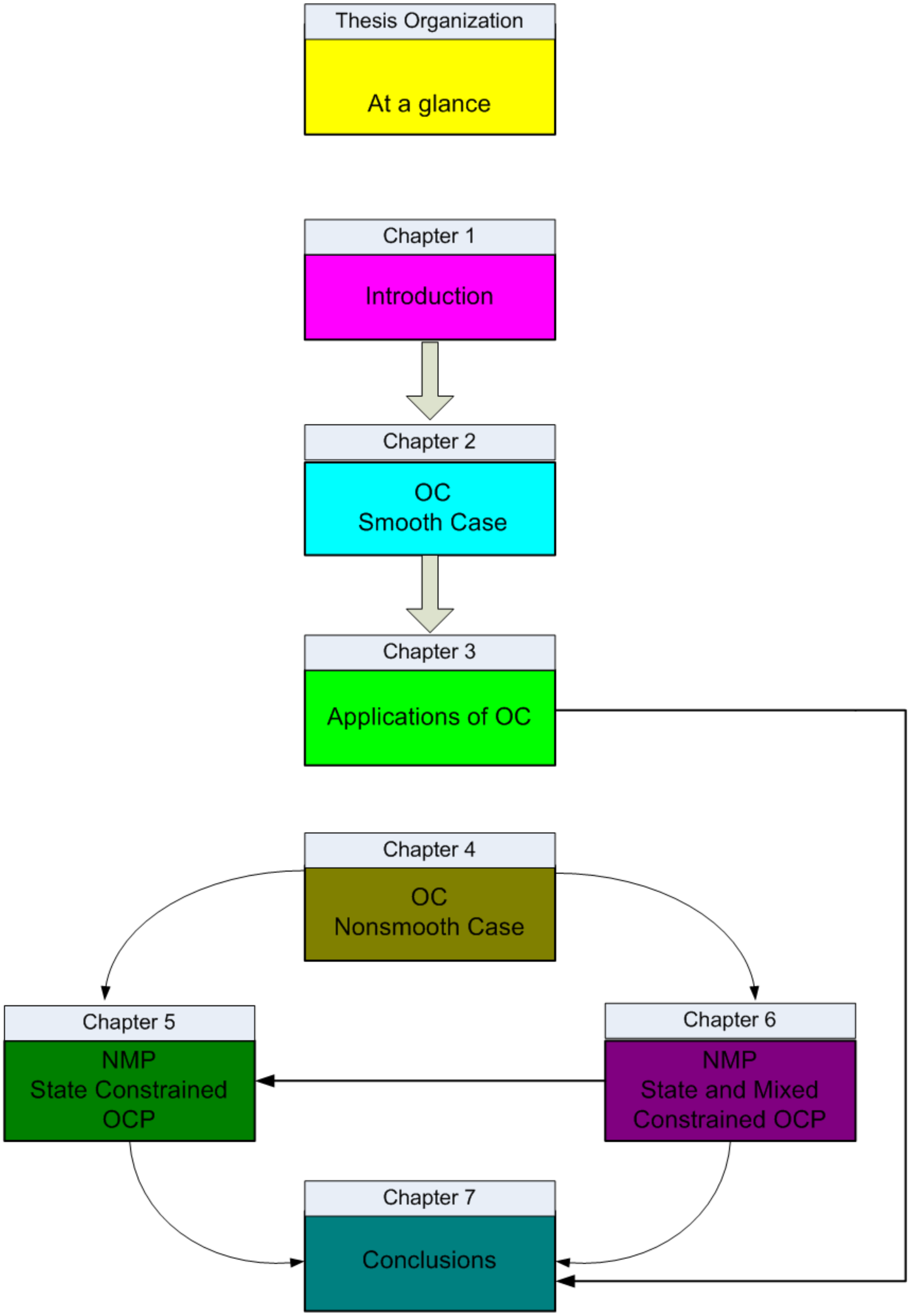
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Chapter 1

Introduction

This dissertation deals with the developments of necessary conditions of optimality (NCO) for optimal control problem (OCP) in the form of maximum principle (MP). The thesis comprises both theory and applications of optimal control problems with constraints.

Necessary Conditions of Optimality (NCO) provide an elegant method to characterize and find solutions to optimal control problems. The main purpose of necessary conditions of optimality is to identify a *small* set of candidates to local minimizers among the overall set of admissible solutions. It is thus the natural interest to construct NCO *as strong as possible* that further reduce the set of identified candidates to a smaller set while still identifying all the local minimizers. The key concept of having such type of minimizers is shown in Figure 1.1.

It is commonly accepted that the Maximum Principle was first proved by L. Pontryagin and collaborators in the late 1950's ([56]). This first version of MPs was derived under *smoothness* assumptions. Later on, a large number of modified, extended and generalized versions of MPs have been developed by several authors and in the late 1970s, extensions in the form of *Nonsmooth Maximum Principles* (NMPs) were proposed by F. Clarke [12].

Necessary Conditions of Optimality for Optimal Control Problems with *State Constraints* have attracted attention since probably the early stage of the control research, specially because of the inter-related applications in diverse engineering fields (for examples, systems and robotics, process engineering and economics, mathematical biology and medicine).

In this thesis we concentrate on optimal control problems with mixed and state constraints in the form of inequalities. We first illustrate the interest of such constraints by introducing them on a previously treated smooth optimal control problem. The aim is to

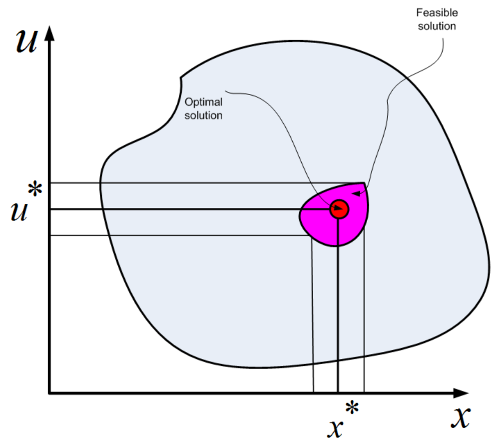

Figure 1.1: Small set of minimizers as optimal solution (x^*, u^*) over all feasible solutions (x, u).

determine vaccination strategies to control the spreading of infectious diseases. We treat the proposed new problems both numerically and analytically. In the second part of this thesis we turn to nonsmooth analysis. We derive new nonsmooth maximum principles in the vein of [19] for problems with both state and mixed constraints.

> As far as our application of interest, we have obtained new control strategy for a general "SEIR" type epidemic disease model by vaccination. We have modified the results in [54] by replacing the so-called *isoperimetric constraint* by a mixed constraint. Our results presented in section 3.2 of Chapter 3 give a realistic control strategy with the limited vaccines. We solve our proposed model both analytically and numerically. We validate the numeric results using information available on the multipliers.
>
> As for theoretic results, we first derive a nonsmooth necessary conditions of optimality for state constrained problems in the form of NMP under some convexity assumptions. These results are then shown to apply when the convexity is removed. Our theoretical work can be seen as a generalization of [19] to problems with state constraints. However, and for technical reasons, we work under assumptions stronger than those in [19]. Nevertheless our results do cover a large class of problems arising in different applications. The main ingredient of these necessary conditions is the "joint" adjoint inclusion with respect to both state and control variables, keeping the feature of being sufficient conditions for *normal linear-convex* problems (see in Chapter 5).

This thesis is organized in the following manner.

In **Chapter 2**, we introduce a review on optimal control problems. We present some essential tools in the form of definitions, lemmas and theorems. We also discuss different forms of optimal control problems arising in literature. Several physical constraints frequently encountered in optimal control problems are discussed here as well as different types of minimizers.

In **Chapter 3**, we briefly mention some optimal control problems arising in different areas. We emphasize that we present a brief description of each problem omitting the detailed analytical as well as numerical discussions. However we refer the reader to the appropriate literature. Our aim is to point out the role of state constraints and/or mixed constraints in applications.

Next we concentrate on applications of optimal control to real problems in mathematical epidemiology of infectious diseases. We study the introduction of constraints in a model proposed in [54]. Our motivation is to study the role of constraints. However we believe that the constraints we propose can be, in some case, a mathematical translation of real constraints. Indeed, if the vaccination interval $[0, T]$ is big enough, the overall limit on the vaccines may not be the best option. Instead it may happen that the number of vaccines available at each instant may be limited or the capability to vaccinate at each unit of time may dictate the need for a constraint on the number of vaccines at each instant. Such considerations led us to propose the replacement of the overall limit of vaccines as proposed in [54] by mixed constraints. Numerical simulations are done using some optimal control solvers and these findings are then partially validated by a theoretical analysis based on maximum principle. Finally we introduce state constraint and, state and mixed constraints separately. In these cases, we only investigate the models numerically omitting the thorough analytical analysis. The numerical simulations for both state constraint and, state and mixed constraints cases are presented in section 3.2.5 of Chapter 3. The results of this chapter have been accepted for publication in MBE [8].

In **Chapter 4**, a brief discussion of nonsmooth analysis and nonsmooth maximum principle for optimal control problems is presented.

Chapter 5 presents a new nonsmooth maximum principle for optimal control problems with state constraints. The *Classical Maximum Principle* (or Pontryagin Maximum Principle) is a necessary condition of optimality for optimal control problems. But for *normal linear-convex* problems, Pontryagin maximum principle (PMP) is a necessary and sufficient condition for optimality.

In the case of *Classical Nonsmooth Maximum Principle*, it is not guaranteed that the optimality conditions are necessary and sufficient because of the nonsmoothness of the problems. An example in this regard can be found in [22]. In order to fix this situation, de Pinho and Vinter came up with necessary conditions of optimality in the "Euler form". The main ingredient of these necessary conditions is the "joint" adjoint inclusion. These necessary conditions are a sufficient condition for normal linear convex problem (see Proposition 4.1 in [22]). However, they fail to be a maximum principle since the *Weierstrass Condition* is not validated. In 2002, de Pinho *et al.* [24] extended the work of de Pinho and Vinter to state constraints and they also showed that such generalization remains a sufficient condition for the normal linear-convex problems. Recently Clarke and de Pinho derived a new nonsmooth maximum principle (see [19]) in the vein of [22].

Our motivation in this chapter is to generalize the result of Clarke and de Pinho [19] to state constrained problems. The NMP derived here is a sufficient condition for normal linear-convex problems. Also our results in state constrained problems are worth in so far as they apply to nonsmooth problems. In fact they coincide with known results for the smooth case. The results of this chapter have been announced in [6].

In **Chapter 6**, we present new nonsmooth maximum principle for optimal control problems with both pure state constraints and mixed constraints [7]. First we have derived a NMP, presented in Proposition 6.4.1, which applies to problems with convex "velocity set", followed by its generalization to nonconvex problems, in the form of Theorem 6.4.2. The novelty of this chapter is to deal with the optimal control problems where both pure state constraints and mixed constraints are present. There are a variety of NMP for constrained problems where either only state constraints have been considered or mixed constraints have been taken into account. One such NMP probably the most successful attempt to cover nonsmooth mixed constrained problems appeared recently in [19] (see also the accompanying paper [20]). However, literature is sparse as far as nonsmooth problems with both such constraints; mixed constraints and pure state constraints.

This motivates us in this chapter to concentrate on problems with both state and mixed constraints in the form of inequalities. We apply "old" techniques developed in [66] and a recent result obtained in [19] to derive nonsmooth necessary conditions for our problem of interest. In so doing we obtain a new NMP for state and mixed constrained problems with the special feature of being a sufficient condition for normal, linear convex problems. To some extent our approach can be viewed as an extension of [24] and [25]. A part of results of this chapter were published in [7]. The complete version of the results have been submitted for publication to ESAIM [5].

Finally in **Chapter 7**, we conclude this thesis by providing a summary of the *present contributions* of our work and some suggestions of *future work*. We pose some open questions concerning state constrained optimal control problems and their applications to the real problems in the field of mathematical biology, mathematical epidemiology and biomedicine to motivate further research.

Chapter 2

Optimal Control: Smooth Case

It is commonly accepted that optimal control theory was born with the publication of a seminal paper by Pontryagin and his collaborators last century, at the end of 1950's. Since then optimal control theory has played a relevant role not only in the dynamic optimization but also in the control and system engineering. Another crucial moment in this theory is closely related with the development of nonsmooth analysis during the 1970's and 1980's. Nonsmooth analysis has triggered a new interest in optimal control problems and thus "brought new solutions to old problems" [67]. Nowadays optimal control theory is essential to different areas like engineering, economics and biology since many problems are modeled as optimal control problems.

The development of optimal control has gained strength by treating multi-variable, time varying systems, as well as many nonlinear problems arising in control engineering. Several authors contributed to the basic foundation of a very large scale research effort initiated in the end of the 1950's, which continues to the present day. The development of Nonsmooth Analysis (see for example, [12] and [67]) has enhanced a wide scope of research as well as it has opened a new horizon in optimal control theory. A challenging area of study in this theory remains that of state constraints. Necessary conditions of optimality for optimal control problems with state constraints have been studied since the very beginning of optimal control theory [56]. In spite of all the recent developments, this subject is far from explored. Throughout this thesis we focus on 'fixed time' optimal control problem with state constraints.

In this chapter, we have discussed some fundamental and basic tools of optimal control problem with and without state constraints only of the smooth case. The optimal control problems with nonsmooth case have been discussed in Chapter 4. All the materials reported here in the form of definitions, lemmas, theorems are available in the existing literatures as mentioned in appropriate citations but will be used extensively in our study.

2.1 Preliminaries and Notations

Before proceeding we need to introduce some definitions and notations that will be used throughout the text.

For g in $\mathbb{R}^m$, inequalities like $g \leq 0$ are interpreted componentwise. Here and throughout this thesis, $\mathbb{B}$ represents the closed unit ball centered at the origin regardless of the dimension of the underlying space and $|\cdot|$, the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$. The *Euclidean distance function* with respect to a given set $A \subset \mathbb{R}^n$ is

$$d_A \colon \mathbb{R}^n \to \mathbb{R}, \qquad y \mapsto d_A(y) = \inf \{|y-x| \,:\, x \in A\}\,.$$

A function $h\colon [a,b] \to \mathbb{R}^p$ lies in $W^{1,1}([a,b];\mathbb{R}^p)$ if and only if it is absolutely continuous; in $L^1([a,b];\mathbb{R}^p)$ iff it is integrable; and in $L^\infty([a,b];\mathbb{R}^p)$ iff it is essentially bounded. The norm of $L^\infty([a,b];\mathbb{R}^p)$ is $\|\cdot\|_\infty$.

The effect of state constraints in the optimal control problems is the introduction of measures as multipliers. These multipliers associated with state constraints are the elements of the topological dual space. The space $C^*([a,b];\mathbb{R})$ is the topological dual of the space of continuous functions $C([a,b];\mathbb{R})$. Elements of $C^*([a,b];\mathbb{R})$ can be identified with finite regular measures on the Borel subsets of [a,b]. The set of elements in $C^*([a,b];\mathbb{R})$ taking nonnegative values on nonnegative-valued functions in $C([a,b];\mathbb{R})$ is denoted by $C^\oplus([a,b];\mathbb{R})$. The norm in $C^\oplus([a,b];\mathbb{R})$, $|\mu|$, coincides with the total variation of μ, $\int_{[a,b]} \mu(ds)$. The support of a measure μ, written as $\operatorname{supp}\{\mu\}$, is the smallest closed set $A \subset [a,b]$ such that for any relatively open subset $B \subset [a,b]\backslash A$ we have $\mu(B) = 0$. More discussions on measures can be found in [67].

2.2 Optimal Control Problems

Since the birth of the optimal control, several authors proposed different basic mathematical formulations of OCPs (fixed time problems). For fixed time problems there are three major mathematical formulations of optimal control problems: *Bolza form*, *Lagrange form* and *Mayer form*.

We start with the general form of *Bolza* problem:

$$(P_B) \quad \begin{cases} \text{Minimize} \;\; l(x(a), x(b)) + \displaystyle\int_a^b L(t, x(t), u(t))dt \\ \text{subject to} \\ \;\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \;u(t) \in U(t) \quad \text{a.e.} \;\; t \in [a, b] \\ \;(x(a), x(b)) \in E. \end{cases}$$

Here $[a, b]$ is a fixed interval. The function $f\colon [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes the system dynamics and $U\colon [a, b] \to \mathbb{R}^m$ is a multifunction. Furthermore, the closed set $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and the functions $l\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $L\colon [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ specify the endpoint constraints and cost. The functional

$$l(x(a), x(b)) + \int_a^b L(t, x(t), u(t))dt \tag{2.2.1}$$

to be minimized is called the *payoff* or *cost* functional. The aim of this problem is to find the pair (x, u) comprising two functions where $u\colon [a, b] \to \mathbb{R}^m$ (the control function) and the corresponding state trajectory x which is an absolutely continuous function $x\colon [a, b] \to \mathbb{R}^n$ (called the state function) satisfying all the constraints of the problem (P_B) and minimizing the cost. A pair (x, u), where x is an absolutely continuous function and u is a function belonging to a certain space $\mathcal{U}$ ($\mathcal{U}$ can be L^1, C, the space of measurable functions, the space of piecewise continuous functions, etc.), such that $\dot{x}(t) = f(t, x(t), u(t))$ a.e., is called a *process*. A '*process*' satisfying all the constraints of the problem (P_B) is called an *admissible process*. We say that (x^*, u^*) is an *optimal solution* if it minimizes the cost over all admissible processes. For optimal control problems one may speak of local or global minimizers. In this thesis we focus on local minimizers. Local minimizers can be of different types as we will see in section 2.4.

If the function $l\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is absent from the cost functional (2.2.1) and all others data remain the same, we obtain the optimal control problem in *Lagrange form*; the cost is simply

$$J(x, u) = \int_a^b L(t, x(t), u(t))dt \tag{2.2.2}$$

On the other hand, if the Lebesgue integrable function $L\colon [a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is absent from the cost functional (2.2.1) and all others constraints remain the same, we obtain the *Mayer form* with cost

$$J(x,u) = l(x(a),x(b)) \tag{2.2.3}$$

However, we can reformulate Bolza form (2.2.1) into Mayer form by *state augmentation*. Let us define,

$$\begin{aligned} &\dot{y}(t) = L(t,x(t),u(t)) \quad \text{a.e.} \\ &y(a) = 0. \end{aligned} \tag{2.2.4}$$

Then the problem (P_B) can be rewritten as following

$$(P_M) \quad \begin{cases} \text{Minimize} \;\; l(x(a),x(b)) + y(b) \\ \text{subject to} \\ \;\dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ \;\dot{y}(t) = L(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ \;u(t)\in U(t) \quad\; \text{a.e.} \;\; t\in[a,b] \\ \;((x(a),x(b)),y(a))\in E\times\{0\}. \end{cases}$$

This new problem (P_M) is now in Mayer form. We refer readers [3, 11, 22, 45, 46] for more extensive studies on the transformations of optimal control problems from the Bolza form to the other two special forms along with their examples.

Different variants of optimal control problems appear over the years. The problems we have mentioned here are fixed time problems (since the time interval $[a,b]$ is fixed). Other problems like free time problems, minimum time problems as well as impulsive control problems and others are out of the scope of this thesis.

2.3 Constrained Optimal Control Problems

Optimal control problems (OCPs) have become challenging because of imposing a great variety of *physical constraints*. These constraints restrict the range of values of both the control and the state variables. When the pathwise constraints are imposed on the state trajectories of the optimal control problems in question, such types of problems are called the *state constrained optimal control problems*. In most cases, such constraints take the form of *scalar functional inequality constraints* because these kinds of state constraints are frequently encountered in engineering applications.

Next we briefly review different constraints usually imposed to optimal control problems in many engineering applications. As we mentioned before, we focus only on constraints for fixed time problems.

The constraints which are usually imposed at the initial point and/or terminal point of a fixed interval $[a, b]$ are called *endpoint constraints*. The most general way of writing this constraint is

$$(x(a), x(b)) \in E. \tag{2.3.5}$$

This includes many types of constraints. Suppose for example, $x(a) = x_a$ and $x(b) \in \mathbb{R}^n$. Then $E = \{x_a\} \times \mathbb{R}^n$.

Endpoint constraints of the form $x(a) = x_a$ and $x(b) \in E_b$ (here E in (2.3.5) is then $E = \{x_a\} \times E_b$) where E_b may be a point or nonempty set are common in applications.

Moreover we can have endpoint constraints in the form of equality or inequality or both. i.e., $\phi_1(x(a), x(b)) = 0$ and/or $\phi_2(x(a), x(b)) \leq 0$. These constraints can be written into the inclusion form (2.3.5) by the right choice of the set E.

The constraints imposed on the control variables of an optimal control problem are called *control constraints*. For example, $u(t) \in U(t)$ is called control constraint, where $u(t)$ takes values in a set $U(t)$. Here $U : [a, b] \to \mathbb{R}^m$ is a multifunction and for $t \in [a, b]$ the value of the multifunction is $U(t)$.

Pathwise constraints encountered in many optimal control problems may be introduced to restrict the range of values taken by functions depending on control and the state variables. Such restrictions can be imposed over the entire time interval $[a, b]$ or any (nonempty) time subinterval. Let us discuss here some common pathwise constraints appearing in the literature. In general, such constraints can be written as

$$(x(t), u(t)) \in C(t) \quad \text{for all } t \in [a, b]$$

where $C : [a, b] \to \mathbb{R}^n \times \mathbb{R}^m$ is a given multifunction. In the literature, however, we encounter explicit constraints of the form:

A. State constraints: In the literature state constraints appear as equality constraints

$$h(t, x(t)) = 0 \quad \text{for all } t \in [a, b],$$

inequality constraints

$$h(t, x(t)) \leq 0 \quad \text{for all } t \in [a, b],$$

or set constraints

$$x(t) \in X(t) \quad \text{for all } t \in [a,b] \tag{2.3.6}$$

where $h\colon [a,b] \times \mathbb{R}^n \to \mathbb{R}^p,\ p \geq 0$ and X is a multifunction $X\colon [a,b] \to \mathbb{R}^n$.

B. Mixed constraint **(also known as state dependent control constraints):** Mixed constraints can be of the form of equalities

$$g(t, x(t), u(t)) = 0 \quad \text{a.e. } t \in [a,b],$$

inequalities

$$g(t, x(t), u(t)) \leq 0 \quad \text{a.e. } t \in [a,b],$$

or both equalities and inequalities, or they may take the general form

$$(x(t), u(t)) \in C(t) \quad \text{a.e. } t \in [a,b] \tag{2.3.7}$$

where $C\colon [a,b] \to \mathbb{R}^q$ is a multifunction.

Observe that in case **A** and **B** constraints in terms of equalities and inequalities can be written as set constraints (2.3.6) and (2.3.7). However, the opposite is also true, i.e., set constraints can be written as equalities (using distance function) or inequalities (using sign distance function).

From the point of view of optimality conditions, state constraints and mixed constraints have different treatments. Necessary conditions for problems with mixed constraints can be obtained when some constraint qualifications (also called in this case regularity conditions) are imposed (see [19]). Such constraint qualifications involve the control variable and they do not make sense when state constraints are present since the state constraints exhibit no dependence on the control variable.

2.4 Types of Local Minimizer

Minimizers are the *solutions* of Optimal Control Problems (OCPs). Minimizers can be *global and local*. Suppose for example, we want to find the minimizers of the problem

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{subject to} \quad x \in \mathbb{R}^n \end{aligned} \tag{2.4.8}$$

Then x_G^* will be a *global minimizer* of (2.4.8), if it minimizes the cost over all other $x \in \mathbb{R}^n$, i.e.

$$f(x_G^*) \leq f(x) \ \ \forall x \in \mathbb{R}^n,$$

and x_L^* will be a *local minimizer* of (2.4.8), if it minimizes the cost over all other x in some neighborhood, i.e. there exists $\varepsilon > 0$ such that

$$f(x_L^*) \leq f(x) \ \ \forall x \in \mathbb{B}(x_L^*; \varepsilon).$$

Let us turn to optimal control problems.

Definition 2.4.1 *An admissible process (x^*, u^*) is a* strong local minimizer *for an optimal control problem if, for some $\varepsilon > 0$, it minimizes the cost over all other admissible processes (x, u) such that*

$$|x(t) - x^*(t)| \leq \varepsilon \quad \textit{for all} \quad t \in [a, b].$$

Definition 2.4.2 *An admissible process (x^*, u^*) for an optimal control problem is called a* weak minimizer *if there exists $\varepsilon > 0$ such that $l(x^*(a), x^*(b)) \leq l(x(a), x(b))$ holds for all process (x, u) satisfying the following conditions:*

$$|x(t) - x^*(t)| \leq \varepsilon \quad \forall t \in [a, b]$$

and

$$|u(t) - u^*(t)| \leq \varepsilon \ \textit{a.e.} \ t \in [a, b].$$

Definition 2.4.3 *An admissible process (x^*, u^*) is a $W^{1,1}$* local minimizer *for an optimal control problem if, for some $\varepsilon > 0$, it minimizes the cost over all other admissible processes (x, u) such that*

$$|x(t) - x^*(t)| \ \leq \varepsilon, \quad \textit{and} \quad \int_a^b |\dot{x}(t) - \dot{x}^*(t)| dt \leq \varepsilon.$$

Also it can be defined as

$$\| \ x - x^* \ \|_{W^{1,1}} \leq \varepsilon.$$

Observe that

$$\| \ x - x^* \ \|_{W^{1,1}} = |x(a) - x^*(a)| + \int_a^b |\dot{x}(t) - \dot{x}^*(t)| dt \leq \varepsilon.$$

Definition 2.4.4 *Let us take a measurable function* $R : [a, b] \to (0, +\infty]$ *called a* radius function. *An admissible process* (x^*, u^*) *is a* local minimizer of radius R *for an optimal control problem if, for some* $\varepsilon > 0$, *it minimizes the cost over all other admissible processes* (x, u) *satisfying*

$$|x(t) - x^*(t)| \ \leq \varepsilon \quad , \quad \int_a^b |\dot{x}(t) - \dot{x}^*(t)| dt \leq \varepsilon,$$

as well as

$$|u(t) - u^*(t)| \leq R(t), \quad a.e. \quad t \in [a, b].$$

A relation between strong and weak local minimizers is that strong local minimizer is always a weak local minimizer but the converse is not necessarily true (see [71]). If (x^*, u^*) is a strong local minimizer, then it is also a $W^{1,1}$ local minimizer. If (x^*, u^*) is a $W^{1,1}$ local minimizer, then it is a local minimizer of radius R (for discussion of such minimizers see [52]).

In this thesis we mainly focus on strong local minimizers and $W^{1,1}$ local minimizers. Local minimizers of radius R (see [14] and [19]) are not treated here since stratified necessary condition are out of scope in our thesis and we work with essentially bounded controls. Suppose we derive a set of necessary conditions for $W^{1,1}$ local minimizers. If we know that (x^*, u^*) is a strong local minimizer, then, since it is also a $W^{1,1}$ local minimizer, (x^*, u^*) should satisfy those necessary conditions.

2.5 The Maximum Principle

The Maximum Principle (MP) provides a set of necessary conditions which should be satisfied by any optimal solution of optimal control problem. Not surprisingly the idea behind derivation of necessary conditions in the form of Maximum Principles is to obtain MPs that produces the smallest set of candidates to the optimal control problems. It is well known that for some problems the classical Maximum Principle is not only a necessary condition of optimality but also a sufficient condition (for a discussion on this feature in a smooth and nonsmooth context see [22]). Here we will present variants of maximum principles both for smooth and nonsmooth optimal control problems with and without state constraints.

The Pontryagin maximum principle (PMP) was proved for the optimal control problems with ‘smooth’ dynamic constraints. This *smooth version* of maximum principles have been the basic foundation of all other extended versions of MPs over the years. To illustrate the smooth maximum principle, we consider the optimal control problem with *state constraints*,

$$(OCP) \quad \begin{cases} \text{Minimize} \;\; l(x(a), x(b)) \\ \text{subject to} \\ \;\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \;h(t, x(t)) \leq 0 \quad \text{ for all } \; t \in [a, b] \\ \;u(t) \in U(t) \quad \text{ a.e. } \; t \in [a, b] \\ \;(x(a), x(b)) \in E. \end{cases}$$

We assume for the time being that the state constraint $h(t, x(t)) \leq 0$ is *absent* from the problem. The smooth maximum principle, which we present now, is valid under smooth assumptions on the data. Here we consider that the functions f , l are all continuously differentiable. Observe that the multifunction U is constant (i.e., $U(t) = U$) and assume U to be a closed set and E is convex and closed. We define the *Pseudo-Hamiltonian* (or *Unmaximized Hamiltonian* or *Pontryagin*) function

$$H(t, x, p, u) = \langle p, f(t, x, u) \rangle.$$

Now the *smooth maximum principle* for the problem (OCP) without state constraints under some appropriate assumptions can be presented in the next Theorem (an adaptation of Theorem 6.2.1 in [67]).

Theorem 2.5.1 ***(The Maximum Principle for (OCP) without State Constraints)**:*
Let (x^*, u^*) be a strong local minimizer for problem (OCP). Then there exist an arc $p \in W^{1,1}([a, b]; \mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$ satisfying
the *Nontriviality Condition* [NT]:

$$(p, \lambda_0) \neq (0, 0),$$

the *Euler Adjoint Equation* [EA]:

$$-\dot{p}(t) = \nabla_x \langle p(t), f(t, x^*(t), u^*(t)) \rangle \;\; \text{a.e.},$$

the global *Weierstrass Condition* [W]:

$$\forall \;\; u \in U(t), \;\; \langle p(t), f(t, x^*(t), u)\rangle \leq \langle p(t), f(t, x^*(t), u^*(t))\rangle \;\; \text{a.e.,}$$

and the *Transversality Condition* [T]:

$$(p(a), -p(b)) = \lambda_0 \nabla l(x^*(a), x^*(b)) + (\eta_1, \eta_2),$$

for some $(\eta_1, \eta_2) \in N_E(x^*(a), x^*(b))$, where $N_E(x^*(a), x^*(b))$ is the normal cone to E.

It is worth mentioning that, since $E \subset \mathbb{R}^n$ is closed and convex, the normal cone to E at $x^* \in E$ is defined by

$$N_E(x^*) = \{\xi \in \mathbb{R}^n : \langle \xi, x - x^*\rangle \leq 0 \;\; \forall x \in E\}.$$

The function p is called the *costate* (or *adjoint*) function and λ_0 the *cost multiplier*. The adjoint equation is also called the costate differential equation.

However, we now turn to the more general case of the same problem (OCP) assuming that the state constraint is imposed. We note that the effect of state constraints is the introduction of *measures* as multipliers. The adjoint multiplier p is then replaced by a function q of *bounded variation* defined by,

$$q(t) = \begin{cases} p(t) + \int_{[a,t)} \gamma(s)\mu(ds) & t \in [a, b) \\ p(t) + \int_{[a,b]} \gamma(s)\mu(ds) & t = b, \end{cases} \tag{2.5.9}$$

where $\gamma : [a, b] \to \mathbb{R}^n$ is a multiplier associated with the state constraints.

Let us assume again that the functions f, l and h are all continuously differentiable as before, that U is a closed set and E is closed and convex. Then the *smooth maximum principles* for the *state constrained optimal control problems* adapted from Theorem 9.3.1 in [67]) reads

Theorem 2.5.2 ***(The Maximum Principle for (OCP) with State Constraints)**:* Let (x^*, u^*) be a strong local minimizer for problem (OCP). Then there exist an arc $p \in W^{1,1}([a, b]; \mathbb{R}^n)$, a scalar $\lambda_0 \geq 0$, $\mu \in C^{\oplus}([a, b])$, and a measurable function $\gamma(t)$: $[a, b] \to \mathbb{R}^n$ such that the following conditions are satisfied:

(i) The Nontriviality Condition [NT]:

$$(p, \mu, \lambda_0) \neq (0, 0, 0)$$

(ii) The Euler Adjoint Equation [EA]:

$$-\dot{p}(t) = \nabla_x \langle q(t), f(t, x^*(t), u^*(t)) \rangle \quad \text{a.e.,}$$

(iii) The Weierstrass Condition [W]:

$$\forall \quad u \in U(t), \quad \langle q(t), f(t, x^*(t), u) \rangle \leq \langle q(t), f(t, x^*(t), u^*(t)) \rangle \quad \text{a.e.,}$$

(iv) The Transversality Condition [T]:

$$(p(a), -q(b)) = \lambda_0 \nabla l(x^*(a), x^*(b)) + (\eta_1, \eta_2),$$

for some $(\eta_1, \eta_2) \in N_E(x^*(a), x^*(b))$,

(v) $\quad \mathrm{supp}\{\mu\} \subset I(x^*)$,

where $I(x^*) := \{t : h(t, x^*(t)) = 0\}$ and q is as in (2.5.9).

We note that the maximum principle stated in Theorem 2.5.2 is of interest only when the state constraint is *nondegenerate*. We omit the discussion of this nondegeneracy issue here as it is out of the scope of this thesis, rather we refer readers to [1, 2, 9, 32, 46, 47, 58, 67] for detailed survey as well as some recent developments on the degeneracy phenomenon.

Chapter 3

Applications of Optimal Control

In this chapter we focus on applications of optimal control arising in different areas. First we give a short overview of some interesting problems in the literature where the introduction of mixed and state constraints are of relevance. We do not discuss such problems since such discussions can be found in the literature. We then concentrate on an optimal control problem proposed in [54] modeling the control of the spread of infection diseases by vaccination involving a well-known "SEIR" (Susceptible, Exposed, Infected and Recovered) compartmental model. For such problem we propose the introduction of mixed constraints. The new problem is then treated numerically using an optimal control solver and their solutions are then compared with the literature. The reason we choose such problem is twofold. First because we believe that mixed and state constraints we suggest are more realistic than those in the literature. And secondly, these problems allow us to illustrate the importance of necessary conditions to validate the numerical solutions.

3.1 Overview of Applied Optimal Control Problems

Van der Pol equation plays a dominating role in our understanding of *nonlinear dynamics* in general by serving as a prototype for various phenomena. It provides an important mathematical model for dynamical systems arising in most of the natural and engineering sciences and also in many physical problems. See for examples [35], [41], [70] for more information, descriptions and solutions of Van der Pol equation.

We present here two models of Van der Pol oscillator where the second model is a variant of such problem. These are fixed interval $[0, t_f]$ and Lagrange type optimal control problems.

In [49] the following Van der Pol oscillator optimal control is proposed:

$$
(\mathrm{P_{VdPs}}) \begin{cases} \text{Minimize } \displaystyle\int_0^{t_f} (x_1^2(t) + x_2^2(t) + u^2(t))\, dt \\ \text{subject to} \\ \quad \dot{x_1}(t) = x_2(t), \\ \quad \dot{x_2}(t) = -x_1(t) + x_2(t)(1 - x_1^2(t)) + u(t), \\ \quad x_1^2(t_f) + x_2^2(t_f) = r^2, \quad [r = 0.2], \\ \quad -0.4 \le x_2(t) \quad \forall\, t \in [0, t_f], \\ \quad u(t) \in [-1, 1] \quad \forall\, t \in [0, t_f], \\ \quad x_1(0) = 1, \quad x_2(0) = 1, \end{cases}
$$

Observe that this problem involves, besides end point constraints and control set constraints, pure state constraint $-0.4 \le x_2(t)$.

The Rayleigh problem is a variant of the Van der Pol oscillator (see [49], [51]).

The optimal control problem for this variant of Van der Pol oscillator in [49] takes the form:

$$
(\mathrm{P_{VdPm}}) \begin{cases} \text{Minimize } J(x, u) = \displaystyle\int_0^{t_f} (x_1^2(t) + u^2(t))\, dt \\ \text{subject to} \\ \quad \dot{x_1}(t) = x_2(t), \\ \quad \dot{x_2}(t) = -x_1(t) + x_2(t)(1.4 - 0.14 x_2^2(t)) + 4u(t), \\ \quad \alpha \le u(t) + \dfrac{x_1(t)}{6} \le 0, \quad \text{for } \alpha = -1, -2 \\ \quad u(t) \in [-1, 1] \quad \forall\, t \in [0, t_f], \\ \quad x_1(0) = -5, \quad x_2(0) = -5, \end{cases}
$$

Observe that now we have mixed constraints of the form

$$
\alpha \le u(t) + \frac{x_1(t)}{6} \le 0.
$$

We refer readers for the detailed analysis and solutions of both the problems (P_{VdPs}) and (P_{VdPm}) which can be found in [49] (see also [51]).

Aerospace and Robotics are two areas where optimal control problems play an important role. The literature is abundant in this respect. So we do not present any of such problems. However we refer the reader to [49] and [55] where an interesting minimum time problem for the planar two link robot is studied.

Before proceeding it is also worth mentioning the research done in [40] (see also [49]) where optimal control is applied to semi-conductor laser. The potential applications of semi-conductor lasers include high-speed optical recording, high-speed printing, long-distance transmission, submarine cable transmission and medical applications along with many others (see [29] for more applications). The dynamics of a standard single-mode laser model is described by a system of ordinary differential equations with two state variables $(S(t), N(t))$ and one control variable $I(t)$.

Optimal control is applied in management and industrial engineering where the state constraints play an influential role in maintaining the performance of the production process. More information and descriptions about such problems can be found in [50] where the authors have discussed the detailed analytical and numerical analysis of the problem both for linear and quadratic cost functional cases.

Finally, we concentrate on application of optimal control in the control of infectious disease. We discuss here the optimal control strategy for the chemotherapy of the infectious HIV model. The mathematical model of HIV is a set of ordinary differential equations which describes the interactions between the $CD4^{+}T$ cells in the human immune systems and the viruses. $CD4^{+}T$ lymphocytes are commonly referred to as *helper- T-cells*. These cells are the main target of the virus. An infected $CD4^{+}T$ cell can produce around 500 new viruses before its death and thus it is more important to destroy them the virus itself [44]. When a free HIV virus enters the body and attacks the *uninfected* $CD4^{+}T$ cells, the cells become infected and go through a *neutral stage* before becoming *actively infected*. The cells in this *latent/interim stage*, which cannot infect other cells are called the *latently infected*. Thus the $CD4^{+}T$ cells are divided into three classes: *active/uninfected* $CD4^{+}T$ cells, whose concentration is represented by a variable $T_A(t)$, and other two types of infected $CD4^{+}T$ cells are *latently infected* and *actively infected* cells with their concentrations represented by $T_L(t)$ and $T_I(t)$ respectively. The concentration of the free infectious virus is represented by $V(t)$.

The influence of the chemotherapy in the HIV model is represented by a *control function* (or *chemotherapy function*) $u(t)$. This control represents the percentage of effect the chemotherapy has on the viral production.

The model under the above descriptions can be presented by the following system of ordinary differential equations:

$$\frac{dT_A}{dt} = \frac{s}{1+V(t)} - \mu_{T_A} T_A(t) + rT_A(t)\Big(1 - \frac{T_A(t)+T_L(t)+T_I(t)}{T_{\max}}\Big) - \mu_i V(t) T_A(t), \quad (3.1.1)$$

$$\frac{dT_L}{dt} = \mu_i V(t) T_A(t) - \mu_{T_L} T_L(t) - \mu_c T_L(t), \quad (3.1.2)$$

$$\frac{dT_I}{dt} = \mu_c T_L(t) - \mu_{T_I} T_I(t), \quad (3.1.3)$$

$$\frac{dV}{dt} = (1-u(t)) N \mu_{T_I} T_I(t) - \mu_i V(t) T_A(t) - \mu_V V(t), \quad (3.1.4)$$

with the initial conditions

$$T_A(0) = T_{A0}, \quad T_L(0) = T_{L0}, \quad T_I(0) = T_{I0}, \quad \text{and} \quad V(0) = V_0. \quad (3.1.5)$$

Here the control class are assumed to be the *measurable functions* defined on the fixed interval $[t_s, t_f]$, with the restriction that $u \in U(t) : 0 \leq u(t) \leq 1, \ \forall t \in [t_s, t_f]$.

In the equations (3.1.1)-(3.1.4) the terms with negative signs and multiplied by the constants μ_{T_A}, μ_{T_L}, μ_{T_I}, and μ_V represent the natural deaths of active $CD4^{+}T$ cells, latently infected $CD4^{+}T$ cells, actively infected $CD4^{+}T$ cells and of the free virus respectively. The details explanations and analysis of the model can be found in [42].

We wish to determine the control policy such that the number of uninfected $CD4^{+}T$ cells should be kept as high as possible while at the same time the negative side-effects and cost of the chemotherapy are minimized. Taking all these into consideration, the objective function in the vein of [42] is:

$$\begin{aligned}
&\text{Minimize} \quad J(u) = \int_{t_s}^{t_f} \Big(-T_A(t) + \frac{1}{2} B u^2(t) \Big) dt \\
&\text{subject to} \\
&\text{the dynamics defined in} \quad (3.1.1) - (3.1.4), \\
&\text{with the initial conditions as in} \quad (3.1.5), \\
&\text{and the control constraints} \quad u \in [0,1] \text{ a.e.}
\end{aligned} \quad (3.1.6)$$

where the parameter $B > 0$ represents the desired ‘weight’ on the benefit and cost.

We refer readers to [42] (see also [49]) for detailed theoretical and numerical analysis as well as the optimal chemotherapeutic strategy of this problem.

3.2 The SEIR Model

This section deals with application of optimal control to a real problem in epidemiology of infectious diseases. we propose modification of optimal control strategy proposed in [54] modeling the control of the spread of infectious diseases by vaccination. We base our analysis on an SEIR model (Susceptible, Exposed, Infectious and Recovered) for the human infectious disease model. The work in this section is part of the paper [8] accepted for publication to MBE.

An effective preventive measure and control of the spreads of infectious disease is of great concern worldwide. Vaccination is assumed to be the most efficient control strategy (provided that such vaccines are properly available in the market) to control the spreading of infectious disease. Since the human body is a highly nonlinear, robust and an adaptive physiological control system, there is a close relationship between control theory and biology. Since the first application of optimal control in biomedical engineering around 1980s [53], several vaccination strategies for infectious diseases of a certain population over a period of time has been successfully modeled as optimal control problems. In this regard, the SEIR–type epidemic model is of importance to study the implication of optimal control for preventive vaccination strategies of certain epidemics. In such model, the targeted population is divided into four classes: susceptible (S), exposed (E), infectious (I), and recovered (immune) (R) class. An SEIR model can represent many human infectious diseases such as measles, pox, flu, dengue, etc. Here we focus on a generic SEIR model, not emphasizing any specific one.

Mathematical models in epidemiology are important tools in analyzing the spread and control of infectious diseases. The mathematical treatment of such models clarify assumptions, variables and parameters, providing new aspects in understanding the spread of diseases and suggesting different control strategies [10]. One of the early models in epidemiology was introduced in 1927 [38] to predict the spreading behaviour of a disease. Since then, many emerging and reemerging infectious epidemic models have been derived. See [28], [57], [63] and [65] for some similar and recent models on such epidemic diseases.

In this section, a new vaccination schedule is performed based on the model proposed by Neilan and Lenhart (2010) [54], where vaccination strategies over a period of time T were studied by means of an optimal control problem based on an "SEIR" model. While the formulation in [54] avoids the introduction of explicit state-space constraints, by adding the penalty term $\int_0^T AI(t)\,dt$ to the cost (called *soft constraints*), we add an explicit bound both on the state and control variables uS by assuming that during the whole vaccination program the total number of vaccines for the susceptible populations will be bounded by $u(t)S(t) \leq V_0$ for all $t \in [0, T]$. This mixed constraint on the vaccinated populations X is introduced taking into account restrictions on the vaccination effect dictated by difficulties associated with vaccines distribution, facilities and lack of qualified persons.

3.2.1 Auxiliary result

Here we state a result that will be of relevance in our development.

Consider the autonomous optimal control problem with mixed state control constraints with a scalar control:

$$
(\mathrm{P}) \quad \begin{cases} \text{Minimize } l(x(0), x(T)) + \displaystyle\int_0^T L(x(t), u(t))\,dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad \text{for all } t, \\ \quad u(t)h(x(t)) + B \leq 0 \quad \text{for all } t, \\ \quad u(t) \in U \quad \text{a.e. } t, \\ \quad x(0) = x_0, \\ \quad x(T) \in \mathbb{R}^n \end{cases}
$$

Here x takes values in $\mathbb{R}^n$, u is a scalar and U is a subset of $\mathbb{R}$. As for the function we have $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$. Let $R > 0$ and set

$$
X := \bar{B}(0, R) \subset \mathbb{R}^n.
$$

Theorem 3.2.1 *Assume that the data of* (P) *satisfies the following conditions:*

E1 *The functions l, L, f, g and h are continuous in x;*

E2 *There exists $M > 0$ such that*

$$
x \in X \implies |g(x)| \leq M(1 + |x|);
$$

E3 *The set U is compact;*

E4 *The function $u \to L(x,u)$ is convex for each $x \in X$ and there exists a constant $\eta > 0$ such that*

$$x \in X,\ u\ \in U \implies L(x,u) \geq \eta.$$

Then if there is at least an admissible process (x, u) such that

$$l(x(0), x(T)) + \int_0^T L(x(t), u(t))\, dt$$

is finite, then (P) *has a solution.*

The proof of above result is a simple adaptation of the proof of Theorem 23.11 in [17].

3.2.2 SEIR Mathematical Model

To model the progress of infectious diseases in a certain population, SEIR models are divided into four different compartments relevant to the epidemic. Those are susceptible (S), exposed (E), infectious (I), and recovered (immune by vaccination) (R).

An individual is in the S compartment if he/she is vulnerable (or susceptible) to catching the disease. Those already infected with the disease but are not able to transmit it are called exposed. Infected individuals capable of spreading the disease are infectious and so in the I compartment and those who are immune are in the R compartment.

Since immunity is not hereditary ([54]), SEIR models assume that everyone is considered to be susceptible to the disease by born. The disease is also assumed to be transmitted to the individual by horizontal incidence, i.e., a susceptible individual becomes infected when in contact with infectious individuals. This contact may be direct (touching or biting) or indirect (air cough or sneeze). The infectious population can either die or recover completely and all those recovered (vaccinated or recovered from infection) are considered immune.

In this four compartmental model, let $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote the number of individuals in the susceptible, exposed, infectious and recovered class at time t respectively. The total population at time t is represented by $N(t) = S(t) + E(t) + I(t) + R(t)$. To describe the disease transmission in a certain population, let e be the rate at which the exposed individuals become infectious, g is the rate at which infectious individuals recover and a denotes the death rate due to the disease in an infected individual. Moreover, let b be the natural birth rate and d denotes the natural death rate. These parameters are assumed constant in a finite horizon of interest. The rate of transmission is described by the number of contacts between susceptible and infectious individuals. If c is the incidence coefficient of horizontal transmission, such rate is $cS(t)I(t)$.

A schematic diagram of the disease transmission among the individuals for an SEIR–type model before any control initiation is shown in Figure 3.1.

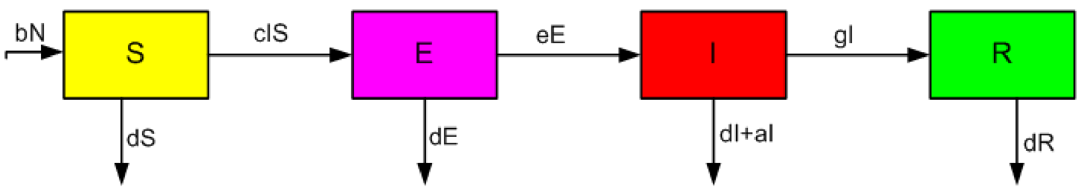

Figure 3.1: SEIR compartmental model before vaccination.

For more information about the model we refer the reader to [36], [54] and references within.

Now we turn to the problem of controlling the spread of the disease by vaccination. Assume that the vaccine is effective so that all vaccinated susceptible individuals become immune. Let $u(t)$ represents the percentage of susceptible individuals being vaccinated per unit of time. Then the vaccinated population can be represented by $u(t)S(t)$ over time and all the vaccinated people goes immediately into the R compartment. Now the diseases transmission takes the modified form of Figure 3.1 as in Figure 3.2.

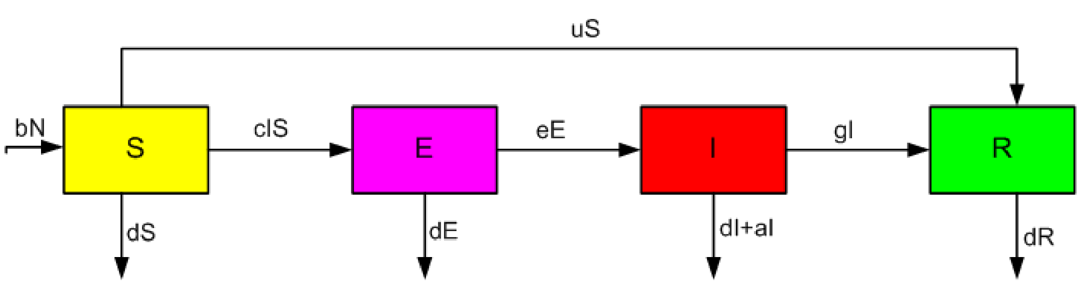

Figure 3.2: SEIR compartmental model after vaccination.

Taking all the above considerations into account and the Figure 3.2 we are led to the following dynamical system:

$$\dot{S}(t) = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t) \tag{3.2.7}$$

$$\dot{E}(t) = cS(t)I(t) - (e+d)E(t) \tag{3.2.8}$$

$$\dot{I}(t) = eE(t) - (g+a+d)I(t) \tag{3.2.9}$$

$$\dot{R}(t) = gI(t) - dR(t) + u(t)S(t) \tag{3.2.10}$$

$$\dot{N}(t) = (b-d)N(t) - aI(t) \tag{3.2.11}$$

with the initial conditions

$$S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ R(0) = R_0,\ N(0) = N_0. \tag{3.2.12}$$

Observe that u acts as the control variable of such system. If $u = 0$, then no vaccination done and $u = 1$ indicates that all susceptible population is vaccinated.

Using the above system Neilan and Lenhart in [54] propose an optimal control problem to determine the vaccination strategy over a fixed vaccination interval $[0, T]$. The idea is then to determine the vaccination policy u so as to minimize the functional

$$J(u) = \int_0^T \big(AI(t) + u^2(t)\big)\, dt. \tag{3.2.13}$$

This means that one seeks to minimize the average number of infectious individuals and the vaccination cost. Indeed, the infectious population may require medical treatment. To accommodate such cost into the objective the right choice of the parameter A should be made; a large A means that the burden of infectious medical care is more important than the vaccination costs. For vaccination costs they use a simple quadratic representation u^2. Also they consider the rate of vaccination taking values in $[0, 0.9]$ instead of $[0, 1]$ to eliminate the case where the entire susceptible population is vaccinated.

The special feature of Neilan and Lenhart model is that they assume that the supply of vaccines is limited. To handle such situation they introduce an extra variable W which denotes the number of vaccines used. Assuming that X denotes the total amount of vaccines used during the whole period of time, the constraint can be represented by

$$\dot{W}(t) = u(t)S(t), \quad W(0) = 0, \quad W(T) = X.$$

Putting all together the problem studied in [54] is then

$$
(P_{NL}) \quad \begin{cases} \text{Minimize } \displaystyle\int_0^T \big(AI(t) + u^2(t)\big)\, dt \\ \text{subject to} \\ \quad \begin{aligned} \dot{S}(t) &= bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \\ \dot{E}(t) &= cS(t)I(t) - (e+d)E(t), \\ \dot{I}(t) &= eE(t) - (g+a+d)I(t), \\ \dot{N}(t) &= (b-d)N(t) - aI(t), \\ \dot{W}(t) &= u(t)S(t), \\ u(t) &\in [0, 0.9] \text{ a.e. } t, \\ S(0) &= S_0,\ E(0) = E_0,\ I(0) = I_0,\ N(0) = N_0,\ W(0) = W_0, \\ W(T) &= X \end{aligned} \end{cases}
$$

Observe that the differential equation for the recovered compartment is not present here. This is due to the fact that the state variable R only appears in the corresponding differential equation and so it has no role in the overall system. Also, the number of recovered individual at each instant t can be obtained from $N(t) = S(t) + E(t) + I(t) + R(t)$. The detailed analysis of ($\mathrm{P_{NL}}$) can be found in [54]. It is worth mentioning that an explicit analytical expression of the optimal control in terms of the state variables and the multipliers is obtained.

3.2.3 Mixed Constrained Model

In some cases, if the vaccination interval $[0, T]$ is big enough, the overall limit on the vaccines may be not the best option. Instead it may happen that the number of vaccines available at each instant may be limited or the capability to vaccinate at each unit of time may dictate the need for a constraint on the number of vaccines at each instant. To deal with this situation we propose a modification of ($\mathrm{P_{NL}}$). We replace the overall limit of vaccines $W(T) = X$ by:

$$
u(t)S(t) \leq V_0 \ \text{ for all } t \in [0, T], \tag{3.2.14}
$$

where V_0 is an upper bound taking values in $\mathbb{R}$. The inequality (3.2.14) is a mixed constraint.

We believe that in some situation the introduction of the mixed constraint may give a more realistic description than that proposed in [54]. The mixed constraint (3.2.14) we now introduce is to be satisfied at every instant of time during the whole vaccination program. We believe that this illustrates a known and major obstacle to the vaccination programs.

With this constraint in hand we now have an optimal control problem with the same cost (3.2.13), the same initial conditions but where the control system consists on the differential equations (3.2.7) –(3.2.11) together with

$$\begin{aligned}
&\dot{W}(t) = u(t)S(t),\\
&u(t)S(t) \leq V_0,\\
&u \in [0,1].
\end{aligned}$$

For the sake of comparison with the results in [54] we opt to keep the same cost functional. However we allow the control rate to take values in $[0,1]$ instead of $[0,0.9]$ as in [54]. This does not affect the analytical treatment of (P_{NL}). As far as the optimal control problem is concerned the differential equation

$$\dot{W}(t) = u(t)S(t)$$

is redundant. In fact the variable W does not appear in the cost, in any other differential equation or in the mixed constraint. So we remove it from our system.

To simplify the forthcoming analysis of our problem it is convenient to rewrite it in the following form:

$$(\mathrm{P}_{\mathrm{mixed}}) \quad \begin{cases} \text{Minimize } \displaystyle\int_0^T L(x(t),u(t))\,dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad \text{a.e. } t, \\ \quad u(t)h(x(t)) - V_0 \leq 0 \quad \text{a.e. } t, \\ \quad u(t) \in [0,1] \quad \text{a.e. } t, \\ \quad x(0) = x_0, \\ \quad x(T) \in \mathbb{R}^n \end{cases}$$

where

$$\begin{aligned}
x(t) &= (S(t), E(t), I(t), N(t)), & L(x,u) &= L_1(x) + L_2(u),\\
f(x) &= f_1(x) + A_1 x, & f_1(x) &= c(-SI, SI, 0, 0),\\
g(x) &= Bx, & h(x(t)) &= \langle C, x(t)\rangle,\\
L_1(x) &= \langle \tilde{A}, x\rangle, & L_2(u) &= u^2
\end{aligned}$$

and

$$\tilde{A} = (0,0,1,0), \qquad C = (1,0,0,0),$$

$$A_1 = \begin{bmatrix} -d & 0 & 0 & b \\ 0 & -(e+d) & 0 & 0 \\ 0 & e & -(g+a+d) & 0 \\ 0 & 0 & -a & b-d \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Our differential equation $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$ is affine in the control and it is nonlinear in the state x due to the term f_1.

Before determining a solution to ($\mathrm{P}_{\mathrm{mixed}}$) we need first to assert that it exists. In this respect Theorem 3.2.1 is of help.

First observe that taking $u = 0$ we get an admissible solution of ($\mathrm{P}_{\mathrm{mixed}}$) with finite cost. Next we construct a set $X \subset \mathbb{R}^4$. This can be done in various ways. One such way is the following. For each constant value of the control u in $[0,1]$, determine the solution $x : [0,T] \to \mathbb{R}^4$ of differential equation

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

that satisfies $x(0) = x_0$. The theory of differential equations asserts that such solution exists in our setting. Let then Z be the set of all values $x(T)$ obtained in this way. The continuity of f and g guarantees that the set Z is bounded and so we can find a $R > 0$ such that $Z \subset \bar{B}(0,R)$. Set then $X = \bar{B}(0,R)$. The conditions **E1**–**E4** of Theorem 3.2.1 hold and thus apply to our problem. This asserts that ($\mathrm{P}_{\mathrm{mixed}}$) has a solution. Let (x^*, u^*) denote such solution. To determine it we now turn to the Maximum Principle.

Define the pseudo Hamiltonian for ($\mathrm{P}_{\mathrm{mixed}}$) as

$$\tilde{H}(x,u,p,q,\lambda) = p \cdot (f(x) + g(x)u) - q \cdot h(x)u - \lambda L(x,u).$$

It is a simple matter to see that the maximum principle for mixed constrained problem given as Theorem 7.1 in [19] applies to such problem. We now seek its consequences. To simplify the notation in what follows we will write $\phi[t]$ to indicate that the function ϕ is evaluated at $x^*(t)$.

Theorem 7.1 in [19] asserts the existence of an absolutely continuous function p, an integrable function q and a scalar $\lambda \geq 0$ such that

(1) $||p||_\infty + \lambda > 0$,

(2) $-\dot{p}(t) = \langle p(t), f_x[t] + g_x[t]u^*(t)\rangle - \langle q(t), h_x[t]u^*(t)\rangle - \lambda L_{1_x}[t]$,

(3) $\mu(t) = \langle p(t), g[t]\rangle - \langle q(t), h[t]\rangle - \lambda L_{2_u}(u^*(t))$ a.e. where $\mu(t) \in N_{[0,1]}(u^*(t))$,

(4) $\forall\ u \in U : h(x^*)u - V_0 \leq 0$,
$\langle p(t), g[t]u^*(t)\rangle - \lambda L_2(u^*(t)) \geq \langle p(t), g[t]u(t)\rangle - \lambda L_2(u(t))$ a.e.,

(5) $\langle q(t), h[t]u^*(t) - V_0\rangle = 0$ and $q(t) \geq 0$ a.e.,

(6) $-p(T) = 0$,

together with the transversality condition

$$p(T) = (0, 0, 0, 0).$$

Furthermore, it is possible to prove the existence of constants K_q^1, K_q^2 such that

$$|q(t)| \leq K_q^1 |p(t)| + K_q^2 \tag{3.2.15}$$

for almost every $t \in [0, T]$.

In **(3)**, $N_{[0,1]}(u^*(t))$ stands for the normal cone from convex analysis to the $[0, 1]$ at the optimal control $u^*(t)$ (see e.g. [12]). The normal cone $N_{[0,1]}(u^*(t))$ is:

$$N_{[0,1]}(u^*(t)) = \begin{cases} 0 & \text{if } u^* \in\ ]0,1[, \\ \mu & \text{if } u^* = 1, \\ -\mu & \text{if } u^* = 0, \end{cases} \tag{3.2.16}$$

for some $\mu \in \mathbb{R}$, $\mu \geq 0$.

We claim that the conditions **(1)**–**(6)** hold with $\lambda \neq 0$. Seeking a contradiction suppose that $\lambda = 0$. Then from **(2)**, we get

$$|\dot{p}(t)| \leq K_1|p(t)| + K_2|q(t)|.$$

This together with (3.2.15) yields

$$|\dot{p}(t)| \leq \tilde{K}|p(t)| + \hat{K}$$

for some $\tilde{K}$ and $\hat{K}$. Appealing to Gronwall's inequality (see [27] and [67]), we can obtain $p(t) = 0$ which is impossible in view of **(1)**. Thus we must take $\lambda > 0$. Scaling the multipliers we can further deduce that **(1)**–**(6)** hold with $\lambda = 1$. In optimal control terms this means that (x^*, u^*) is normal. We have $\tilde{H}_u(x^*(t), u^*(t), p(t), q(t), 1) = 0$. In this case, we also have

$$\tilde{H}_{uu}(x^*(t), u^*(t), p(t), q(t), 1) = -2. \tag{3.2.17}$$

We now turn to show the consequences of **(1)**–**(6)** with $\lambda = 1$. Let us write $p(t) = (p_s(t), p_e(t), p_i(t), p_n(t))$ and $x^*(t) = (S^*(t), E^*(t), I^*(t), N^*(t))$. Then equations **(1)**–**(6)** with $\lambda = 1$ reduce to

(1') $||p||_\infty + 1 > 0$,

(2') $-\dot{p}_s(t) = -dp_s(t) - cI^*(t)p_s(t) - u^*(t)p_s(t) + cI^*(t)p_e(t) - q(t)u^*(t)$,
$-\dot{p}_e(t) = -(e+d)p_e(t) + ep_i(t)$,
$-\dot{p}_i(t) = -cS^*(t)p_s(t) + cS^*(t)p_e(t) - (g+a+d)p_i(t) - ap_n(t) - A$,
$-\dot{p}_n(t) = bp_s(t) + (b-d)p_n(t)$,

(3') $\mu(t) = -S^*(t)p_s(t) - q(t)S^*(t) - 2u^*(t))$ a.e. where $\mu(t) \in N_{[0,1]}(u^*(t))$,

(4') $\forall\ u \in [0,1] : S^*(t)u \leq V_0$,
$-p_s(t)u^*(t)S^*(t) - u^{*2}(t) \geq -p_s(t)u(t)S^*(t) - u^2(t)$ a.e.,

(5') $q(t)(u^*(t)S^*(t) - V_0) = 0$ and $q(t) \geq 0$ a.e.,

(6') $p_s(T) = p_e(T) = p_i(T) = p_n(T) = 0$,

where

$$\begin{aligned}&\tilde{H}(x, u, p_s, p_e, p_i, p_n, q, 1) = p_s\big(bN(t) - dS(t) - cS(t)I(t) - u(t)S(t)\big)\\ &+p_e\big(cS(t)I(t) - (e+d)E(t)\big) + p_i\big(eE(t) - (g+a+d)I(t)\big) + p_n\big((b-d)N(t) - aI(t)\big)\\ &-q(u(t)S(t) - V_0) - (AI(t) + u^2(t)).\end{aligned}$$

Since $u^*(t) \le \dfrac{V_0}{S^*(t)}$, we can deduce that for reasonable values of V_0 and $S^*(0)$, we always have $\dfrac{V_0}{S^*(t)} < 1$. So we have $u^*(t) \in \left[0, \dfrac{V_0}{S^*(t)}\right]$. Now, if the mixed constraint is active, then

$$u^*(t) = \frac{V_0}{S^*(t)} \text{ and } u^*(t) \neq 0.$$

If the mixed constraint is inactive, we get $q(t) = 0$. If moreover $u^*(t) \neq 0$, then $\mu(t) = 0$ in **(3')** and consequently

$$u^*(t) = -\frac{p_s(t)S^*(t)}{2}.$$

We conclude from the above that

$$u^*(t) = \max\left\{0, \min\left\{\frac{V_0}{S^*(t)}, -\frac{p_s(t)S^*(t)}{2}\right\}\right\}. \tag{3.2.18}$$

We emphasize that u^* is indeed the unique optimal control of our problem. In fact, we have established that (P_{mixed}) has a solution. Any solution of our problem must satisfy the Maximum Principle. However the only control satisfying the conclusions of the Maximum Principle is u^* defined by (3.2.18). So u^* thus defined is the optimal solution to our problem.

It is worth mentioning that as in [54] we obtain a closed form for the optimal control of (P_{mixed}). It is of importance to note that if the mixed constraint is active and the control takes values in $]0, 1[$, then we get from **(3')** that $q(t) = -p_s(t) - \dfrac{2V_0}{S^{*2}(t)}$. These information is of importance to check the numerical solution as we will see.

3.2.4 State Constrained Model

In addition to the mixed constrained case, we also investigate the model introducing state constraint and state and mixed constraints separately. For these two cases, we only study the models numerically keeping the details analytical investigations for future work. The motivation of introducing state constraint is that, since the spreading of the disease is given by $cS(t)I(t)$ it is however reasonable to keep an upper bound on the number of susceptible individuals. Recall that all people are susceptibles by born, so it is reasonable to keep the number of susceptible individuals as lower as possible from being infected by the virus and/or infectious individuals. This can be done by an efficient vaccination because after vaccination, a susceptible individual becomes immune. So the idea is that an upper bound on the number of susceptible individuals can be of help to keep the susceptible individuals lower. The translation in mathematical terms of the upper bound on the number of susceptible individuals is the state constraint

$$h(x(t)) \le S_{max}.$$

Thus, with the notation of the previous section, we propose the optimal control problem

$$(\mathrm{P}_{\text{state}}) \quad \begin{cases} \text{Minimize } \displaystyle\int_0^T L(x(t), u(t))\, dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad \text{a.e. } t, \\ \quad h(x(t)) \le S_{max} \quad \text{for all } t, \\ \quad u(t) \in [0,1] \quad \text{a.e. } t, \\ \quad x(0) = x_0, \\ \quad x(T) \in \mathbb{R}^n \end{cases}$$

It is well known that the analysis of the Maximum Principle for optimal control problems with state dependent is quite challenging due to the presence of measures as multipliers associated with the state constraint. Although our numerical simulation shows quite better control strategy, but the analytical analysis may be harder. In this respect, [7], [64] or [67] can be of help for such analytical treatment.

Taking into account that the vaccination effort is at each instant bounded by the resources we also investigate another problem, this one with both mixed and state constraints. That problem is

$$(\mathrm{P}_{\text{SM}}) \quad \begin{cases} \text{Minimize } \displaystyle\int_0^T L(x(t), u(t))\, dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad \text{a.e. } t, \\ \quad h(x(t)) \le S_{max} \quad \text{for all } t, \\ \quad u(t)h(x(t)) \le V_0 \quad \text{a.e. } t, \\ \quad u(t) \in [0,1] \quad \text{a.e. } t, \\ \quad x(0) = x_0, \\ \quad x(T) \in \mathbb{R}^n \end{cases}$$

This problem reflects the need to find vaccination strategies with a maximum number of vaccines V_0 at each instance that would keep the number of susceptible individuals low. As with (P_{state}) we omit here any analytical study of the solution of (P_{SM}); we only discuss the results of numerical simulations, a task conducted in the next subsection.

3.2.5 Numerical Solutions and Discussions

In this section, we shall solve the new problem numerically. We also compare the solution of $(\mathrm{P}_{\mathrm{NL}})$ and $(\mathrm{P}_{\mathrm{mixed}})$.

We performed numerical simulations to obtain the optimal vaccination schedules for our model in different scenarios. To do these simulations we used the Imperial College London Optimal control Software – ICLOCS – version 0.1b [31]. ICLOCS is an optimal control interface, implemented in Matlab, for solving the optimal control problems with general path and boundary constraints and free or fixed final time. ICLOCS uses the IPOPT – Interior Point OPTimizer – solver which is an open-source software package for large-scale nonlinear optimization [68].

Considering a time interval of 20 years, a time-grid with 10000 nodes was created, that is, for $t \in [0, 20]$ we get $\Delta t = 0.002$. We use the parameters of [54] and we present them in Table 3.1. Since we used a direct method and consequently, an iterative approach, we imposed an acceptable convergence tolerance at each step of $\varepsilon_{rel} = 10^{-9}$.

Table 3.1: Parameters and constants with their values [54].

Parameters and Constants	Definition of Parameters	Clinical values
b	natural birth rate	0.525
d	natural death rate	0.5
c	incidence coefficient	0.001
e	exposed to infectious rate	0.5
g	recovery rate	0.1
a	disease induced death rate	0.2
A	weight parameter	0.1
T	number of years	20
S_0	initial susceptible population	1000
E_0	initial exposed population	100
I_0	initial infected population	50
R_0	initial recovered population	15
N_0	initial population	1165

We solved both problems ($\mathrm{P_{NL}}$) and ($\mathrm{P_{mixed}}$) and we present the results of our analysis in several steps. First we solve the problem ($\mathrm{P_{NL}}$) in absence of control measures (i.e., we consider $u = 0$) and the results are presented in Figure 3.3.

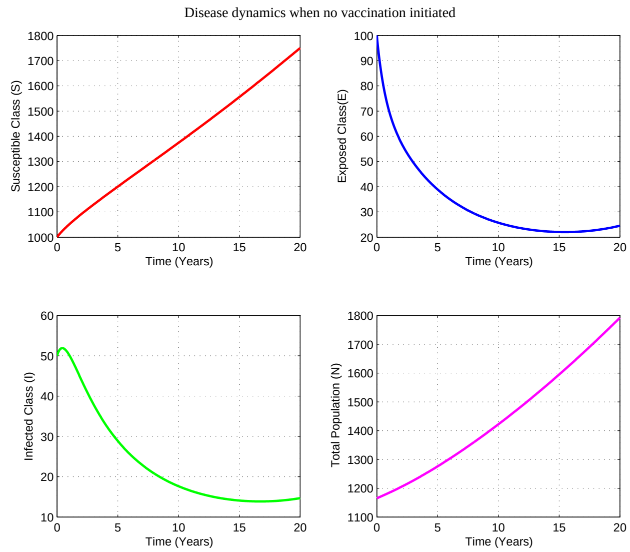

Figure 3.3: The disease dynamics without vaccination during the epidemic.

These results show that without any preventive control the susceptible individuals continue to grow up. Also the number of infectious individuals increases in the very beginning of the time interval achieving a maximum value before $t = 5$ and steadily decreases until it starts to slowly increase in the last instants of the interval. We next solve the optimality systems of the problem ($\mathrm{P_{NL}}$) considering the *unlimited vaccines*, meaning that we can vaccinate the susceptible individuals as many as we choose during the 20 years. The results obtained in this case are shown in Figure 3.4.

Now we solve ($\mathrm{P_{NL}}$) but now taking the vaccines limitation into account. As in [54], we consider that at the end of the 20 years program no more than 2500 susceptible individuals may be vaccinated and the results obtained in this case are presented in Figure 3.5.

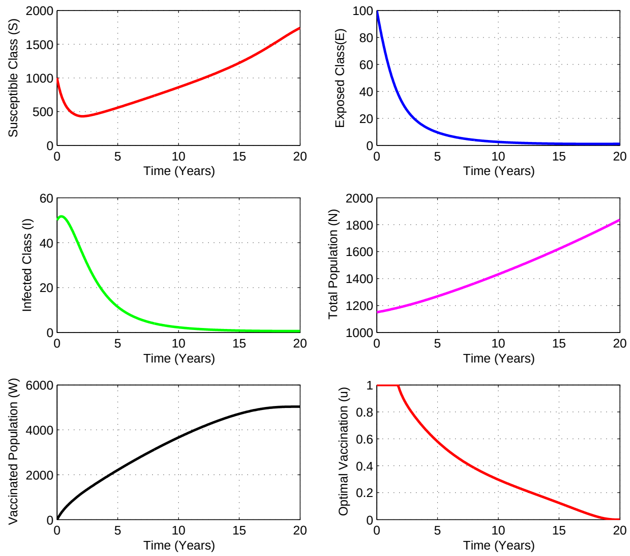

Figure 3.4: The optimal trajectories and optimal vaccination rate without constraint (i.e., unlimited vaccines).

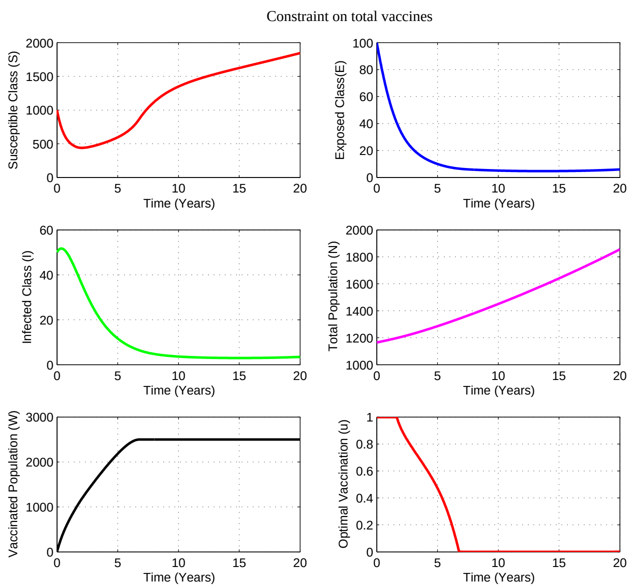

Figure 3.5: The optimal trajectories and optimal vaccination rate with constraint (i.e. limited vaccines with $W(20) \leq 2500$).

Mixed Constraints:

Now we solve our proposed model for the optimality systems of the problem (P_{mixed}) taking the *mixed constraints* into account with $V_0 = 125$. The results for the optimal vaccination schedule are presented in Figure 3.6.

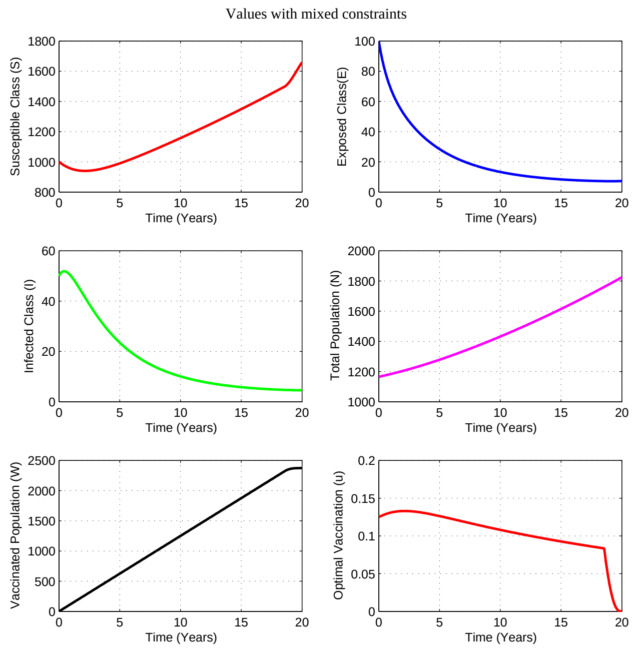

Figure 3.6: The optimal trajectories and optimal vaccination rate with mixed constraints (i.e., $V_0 = 125$).

In Figure 3.6, we can see that in our mixed constrained model the vaccinated people remain below 2500 at the end of the vaccination program, although we do not impose any constraint on the total number of vaccines.

On the other hand, in Figure 3.5 we see that the vaccinated people have been limited by 2500 at the end of program. In contrast to the two previous cases the control variable (rate of vaccination) never achieves its maximum (i.e., it is always less that 1). This is not due to any smoothing effect of the mixed constraint. In fact, the reduced number of vaccines at each instant prevents the use of the maximum rate of vaccination. As illustrated in Figure 3.9 (bellow) the optimal control dictates that the mixed constraint should remain active during almost all the period of vaccination; it only becomes inactive in the very end, dropping to zero at $t = 20$. The lack of vaccines available is also responsible for a cost slightly higher than that obtained before (see Table 3.2).

Although our results with mixed constraints are not as good as the other two cases, they may be more realistic in some situations where the vaccination process is restricted.

Next we focus on the numerical solution of the (P_{mixed}) problem. ICLOCS uses IPOPT and so we have access to the multipliers for such problem. This means we get retrieve the numerical values of $p_s,\ p_e,\ p_i,\ p_n$ and q of the necessary conditions. Those values are presented in Figure 3.7.

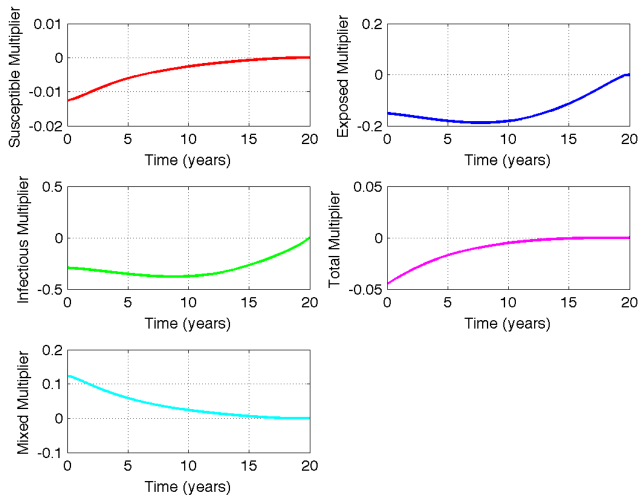

Figure 3.7: The computed adjoint multipliers and the mixed constraint multiplier for (P_{mixed}).

To test the validity of our results, we did the following steps:

- We compare the numerical control with the analytical control given by (3.2.18). As seen in Figure 3.8, we have a match between those two.
- We compare the numerical values of the mixed multiplier q with the analytical q given by
$$q(t) = -p_s(t) - \frac{2V_0}{(S^*(t))^2}. \tag{3.2.19}$$

Recall that when the mixed constraint is inactive, then $q(t)$ should be zero. Such comparison is illustrated in Figure 3.9. Those graphs confirm our analysis and give us some guarantee of the validity of our solution. A more complete analysis could be done using second order sufficient conditions. However, that falls out of the scope of this work. More details can be found in [8].

State Constraint Case:

Finally we present the numerical simulations of (P_{state}). We recall that we do not present here any analytical analysis. The results are purely numerical and a thorough study

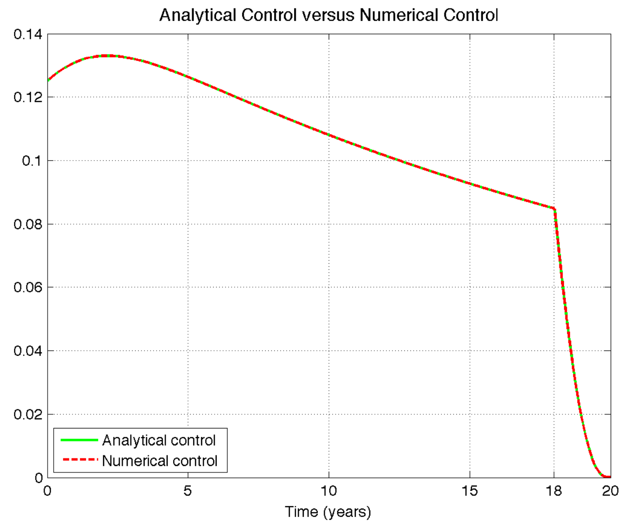

Figure 3.8: Computed or numerical control and the "analytical" control for (P_{mixed}).

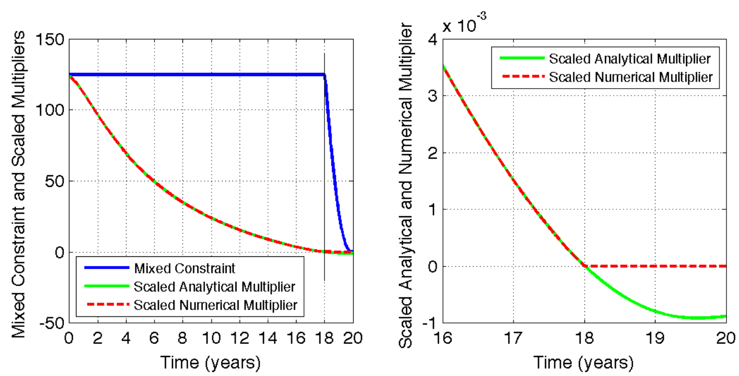

Figure 3.9: The mixed constraint together and the (scaled) analytical (satisfying (3.2.19)) and computed mixed multipliers for (P_{mixed}).

is needed. We take $S_{max} = 1100$. The simulations are presented in Figure 3.10. A remarkable feature of these results is that by constraining S, we are able to keep the susceptible individuals lower and thus the infectious individuals lower at the end of a certain vaccination program. This approach also suggests a cost effective control strategy, but this approach may need a very high number of vaccines available for such program. Nevertheless, for some infectious diseases (e.g., measles) this may be a realistic scenario

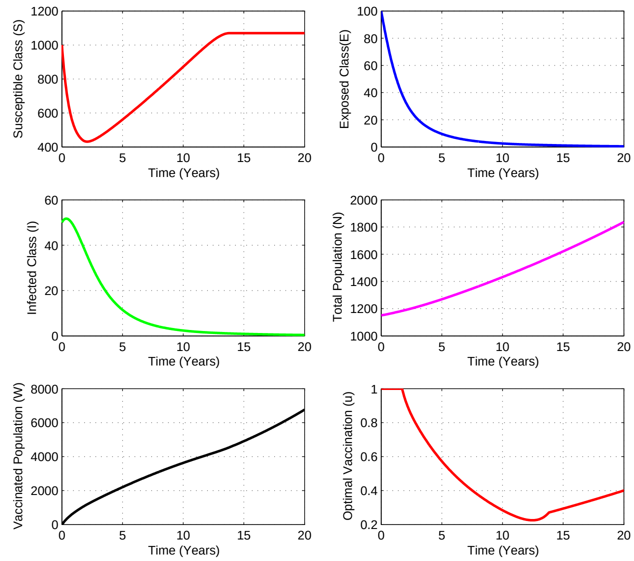

Figure 3.10: The optimal trajectories and optimal vaccination rate with state constraint (i.e., $S_{max} = 1100$).

Table 3.2: Summary of cost functionals and types of controls.

Cases	Values of Costs	Control types
Without control	46.49	n/a
Unlimited vaccines	24.13	$1 - s - 0$
Limited vaccines	25.37	$1 - s - 0$
Mixed constraint	33.68	$s - 0$
State constraint	24.77	$1 - s$
State and mixed constraints	25.57	singular

since there have available vaccines for people.

We now present the numerical results for (P_{SM}). Recall that the purpose of numerical simulations is to check if it is worth to overtake an extended study of such problems. Here, as before, we take $S_{max} = 1100$ and after some preliminary simulations we opt to consider $V_0 = 400$. Our findings are presented in Figure 3.11.

The values of the cost functionals and types of controls are summarized in Table 3.2.

A comparison of the evolution of infected people is shown in Figure 3.12.

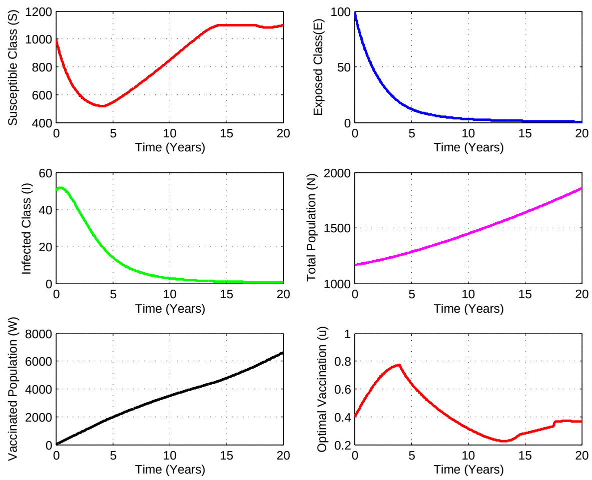

Figure 3.11: The optimal trajectories and optimal vaccination rate with state and mixed constraints (i.e., $V_0 = 400$, $S_{max} = 1100$).

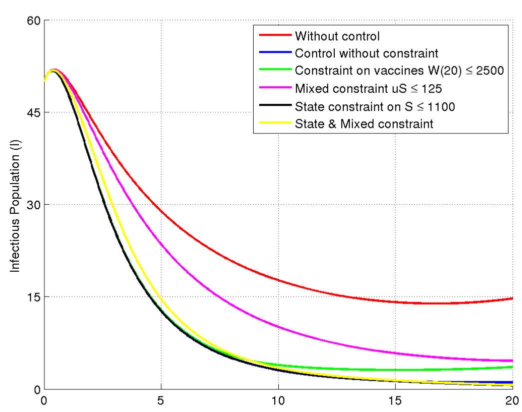

Figure 3.12: A comparison of the evolution of infected population over time.

Chapter 4

Optimal Control: Nonsmooth Case

Nonsmooth Analysis deals with the local approximation of non-differentiable functions and of sets with non-differentiable boundaries; consequently it can be treated as the branch of *Nonlinear Analysis* [13]. In the last few decades this field has grown up because of the recognition of nondifferentiable phenomena. The field of nonsmooth optimization is significant, not only because of the existence of non-differentiable functions arising directly in applications, but also because several important methods for solving difficult smooth problems lead directly to the need to solve nonsmooth problems, which are either smaller in dimension or simpler in structure. See for examples ([12], [13] and [67]) for more details about the background and importance of nonsmooth analysis in optimal control theory.

There has been a sustained and fruitful interaction between Nonsmooth Analysis and Optimal Control. Nonsmooth analysis is an important tool in optimal control theory.

4.1 Nonsmooth Analysis

In the *classical* sense, derivatives of a function f are related to normal vectors to tangent hyperplanes; for any differentiable function f the vector $(f'(x), -1)$ is a downward normal to the graph of f at $(x, f(x))$. The graph of f is defined by $\mathrm{Gr} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha = f(x)\}$. This geometric relationship is the key for the development of nonsmooth analysis. Instead of considering derivatives as elements of normal subspaces to smooth sets, 'generalized derivatives' are defined to be elements of normal cones to possibly nonsmooth sets. Let us start with some definitions that will be useful to introduce nonsmooth calculus.

Let $A \subset \mathbb{R}^n$ be a nonempty closed set with $x \in \mathbb{R}^n \backslash A$. We call y the *closest point* in A or

$Proj_A(x)$ (i.e. the projection of x onto A) (see Figure 4.1) if $y \in \text{Proj}_A(x)$ such that

$$\| x - y' \| \geq \| x - y \|, \ \forall y' \in A$$

which is equivalent to write

$$\langle \omega, y' - y \rangle \leq \sigma \| y' - y \|^2, \ \forall y' \in A \ \text{ and some } \sigma > 0,$$

where the vector $\omega = x - y$ is perpendicular to A at y.

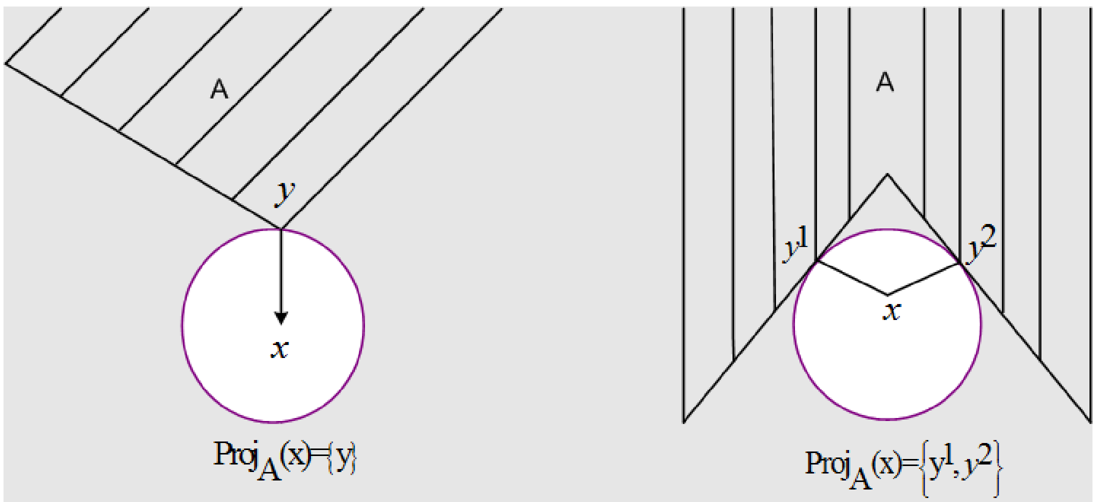

Figure 4.1: Geometrical interpretation of proximal normal and limiting normal cones (source: [23]).

Any nonnegative multiple $\zeta = t\omega, \ t > 0$ of ω is called a *proximal normal* vector (see [67]. That is, a *vector* ζ is called a *proximal normal* to A at y iff for some $\sigma > 0$ the following *proximal normal inequality* holds:

$$\langle \zeta, y' - y \rangle \leq \sigma \| y' - y \|^2, \ \forall y' \in A.$$

The set of all such vectors, which is a convex cone [19] containing 0 is denoted by $N_A^P(y)$ and is referred to as the *Proximal Normal Cone.*

A vector ζ is called the *limiting normal to A at y* if for each $i \in \mathbb{N}$,

$$\zeta = \lim \zeta_i, \ \forall \zeta_i \in N_A^P(y_i), \ y_i \in A, \ y_i \to y,$$

and the set of all such limiting normals, denoted by $N_A^L(y)$ is a cone, called the *Limiting Normal Cone* to A at y.

Given a lower semicontinuous function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $x \in \mathbb{R}^n$ where $f(x) < +\infty$ such that $\text{dom} f = \{x : f(x) < +\infty\}$, the *proximal subdifferential* (or set of all *proximal subgradients*) of f at $x \in \text{dom} f$ is defined as the set

$$\partial^P f(x) := \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N^P_{\text{epi} f}(x, f(x))\}.$$

where $\text{epi} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq f(x)\}$ denotes the epigraph of a function f. The *limiting subdifferential* (or set of all *limiting subgradients*) of a function f at $x \in \text{dom} f$ denoted by $\partial^L f(x)$ is obtained by the set

$$\partial^L f(x) := \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N^L_{\text{epi} f}(x, f(x))\}.$$

However, the nonsmooth calculus can be developed via the theory of *generalized gradients* in the context of *locally Lipschitz function.* If a function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz near x, then the *generalized gradients* $\partial^C f(x)$ coincides with co $\partial^L f(x)$ (*convex hull* of $\partial^L f(x)$); also in similar fashion, the associated normal cone $N_A^C(x)$ to a set A at a point x coincides with $\overline{\text{co}}\, N_A^L(x)$. This *generalized gradients* and its calculus were first defined by Clarke in 1973 [12], so $\partial^C f(x)$ and $N_A^C(x)$ are also called the *Clarke subdifferential* and the *Clarke normal cone* respectively. For more details on such nonsmooth analysis concepts and generalized gradients as well as its basic calculus, we refer readers for example to [12, 13, 52, 67].

4.2 Nonsmooth Maximum Principle

The *nonsmooth maximum principles* have been one of the central attractions of the nonsmooth optimization problems for a long time. In the 1970s F. Clarke generalized the convex subdifferentials of Rockafellar to cover Lipschitz continuous functions and to some extent, lower semi-continuous functions (see, for example [12]). He also successfully applied nonsmooth analysis to optimization and optimal control theory. In 1976s Mordukhovich proposed the concept of limiting subdifferential and he showed how transversality conditions in the nonsmooth maximum principle could be weakened. We now discuss here the *nonsmooth maximum principle* for optimal control problems with state constraints. We consider again our problem (OCP) with state constraints, but we will impose the following hypotheses which make reference to an optimal solution (x^*, u^*) and a parameter $\varepsilon > 0$:

(NH1): The function $(t, u) \to f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}$ measurable and there exist $\varepsilon > 0$ and an integrable function $k(t)$ such that, for almost every $t \in [a, b]$ the following condition holds:

$$|f(t, x_1, u) - f(t, x_2, u)| \leq k(t)\|x_2 - x_1\|, \ \forall u \in U(t), \ (x_1, x_2) \in \mathbb{B}(x^*, \varepsilon).$$

(NH2): l is Lipschitz near $(x^*(a), x^*(b))$ with Lipschitz constant K_l.

(NH3): h is upper semicontinuous and for each $t \in [a, b]$ the function $h(t, \cdot)$ is Lipschitz on $x^*(t) + \mathbb{B}(0, \varepsilon)$ with Lipschitz constant K_h.

(NH4:) GrU is a Borel set,

where GrU is the graph of the multifunction $U : [a, b] \to \mathbb{R}^m$ defined by

$$GrU := \{(t, u) \in [a, b] \times \mathbb{R}^m : u \in U(t)\}.$$

Theorem 4.2.1 ***(The Nonsmooth Maximum Principle for (OCP) with State Constraints*** *(Theorem 9.3.1 in [67]))* Let (x^*, u^*) be a strong local minimizer for problem (OCP) with state constraints and assume that hypotheses (NH1)–(NH4) are satisfied. Then there exist an arc $p \in W^{1,1}([a, b]; \mathbb{R}^n)$, a scalar $\lambda_0 \geq 0$, $\mu \in C^{\oplus}([a, b])$, and a measurable function $\gamma(t) : [a, b] \to \mathbb{R}^n$ satisfying $\gamma(t) \in \partial_x^{>} h(t, x^*(t)) \ \ \mu - a.e.$ such that the following conditions are satisfied.

(i) The Nontriviality Condition [NT]:

$$(p, \mu, \lambda_0) \neq (0, 0, 0),$$

(ii) The Euler Adjoint Equation [AE]:

$$-\dot{p}(t) \in \partial_x^C \langle q(t), f(t, x^*(t), u^*(t)) \rangle,$$

(iii) The Weierstrass Condition [W]:

$\forall \ \ u \in U(t)$,

$$\langle q(t), f(t, x^*(t), u) \rangle \leq \langle q(t), f(t, x^*(t), u^*(t)) \rangle \ \ \text{a.e.},$$

(iv) The Transversality Condition [T]:

$$(p(a), -q(b)) \in \lambda_0 \partial l(x^*(a), x^*(b)) + (\eta_1, \eta_2),$$

for some $(\eta_1, \eta_2) \in N_E^C(x^*(a), x^*(b))$,

(v) $\quad \text{supp}\{\mu\} \subset I(x^*)$,

where $I(x^*) := \{t : h(t, x^*(t)) = 0\}$, q is as in (2.5.9), and the partial subdifferential $\partial_x^>$ is defined by

$$\partial_x^> h(t,x) := \text{co}\,\{\gamma : \exists (t_i, x_i) \xrightarrow{h} (t,x) : h(t_i, x_i) > 0\ \forall i,\ \nabla_x h(t_i, x_i) \to \gamma\}. \tag{4.2.1}$$

Several extended versions, and even more strengthened forms of nonsmooth maximum principles for state constrained optimal control problems have been developed over the years. We refer readers [12, 15, 16, 21, 66] for the detailed presentations and to [18, 19] for the recent developments in the nonsmooth maximum principle.

Chapter 5

Nonsmooth Maximum Principle for State Constrained Problems

In this chapter we shall present the first theoretical result of this thesis. It is a *Nonsmooth Maximum Principle* (NMP) for state constrained problems and its presentation breaks into two: the Nonsmooth Maximum Principle is first proved in the convex case and then convexity is removed. These results on state constrained problems have been announced in [6].

State constrained optimal control problems have been the focus of extensive research in the optimal control theory since the very beginning of the Pontryagin maximum principle (PMP) in 1956 [56]. Recently a special attention has also been paid to phenomena associated with the problems like nondegeneracy, normality and regularity of minimizers; see, for example [1], [4], [32], [33], [34], to name but a few. Research for nonsmooth problems was triggered by the seminal paper [66]. Initial version of NMP (see [12]) failed to be sufficient for linear convex problems in the normal form. One of the first successful attempts to derive nonsmooth necessary optimality conditions with such feature was proposed in [22] and generalized to state constrained problems in [24] and [25]. Regrettably those necessary conditions did not include the Weierstrass condition responsible for the very name Maximum Principle. More recently the setbacks in [22] were taken care of by Clarke and de Pinho in [19] where a new variant of the nonsmooth maximum principle is derived by appealing to [15]. As in [22], Lipschitz continuity of dynamics with respect to both state and control is assumed, the special ingredient responsible for sufficiency of the nonsmooth maximum principle when applied to normal linear convex problems.

In this chapter, we generalize the result of Clarke and de Pinho [19] (see also [18]) to state constrained problems.

We obtain a new variant of the nonsmooth maximum principle, improving on [25] by adding the Weierstrass condition to the previous conditions while keeping the interesting feature of being a sufficient condition for normal linear-convex problems. Our approach follows closely that of [24] and [25].

5.1 Problem Statement and Assumptions

We consider the optimal control problem

$$(P) \quad \begin{cases} \text{Minimize } l(x(a), x(b)) + \displaystyle\int_a^b L(t, x(t), u(t))\ dt \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \ h(t, x(t)) \leq 0 \quad \text{ for all } \ t \in [a, b] \\ \ u(t) \in U(t) \quad \text{ a.e. } \ t \in [a, b] \\ \ (x(a), x(b)) \in E. \end{cases}$$

Our problem (P) involves measurable control functions $u\colon [a, b] \to \mathbb{R}^m$ which satisfies the control constraints $u(t) \in U(t)$ and absolutely continuous function $x\colon [a, b] \to \mathbb{R}^n$. The function describing the dynamics is $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. Moreover h and L are scalar functions $h : [a, b] \times \mathbb{R}^n \to \mathbb{R}$, $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, U is a multifunction, $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and the interval $[a, b]$ is fixed.

We recall that in the absence of the state constraint $h(t, x(t)) \leq 0$ the problem (P) is referred to as a *standard optimal control problem* and we shall denote such problem by (S). A pair (x, u) is called an admissible process if it satisfies the constraints of the problem (P) (or (S)) with finite cost and the pair (x^*, u^*) will always denote the solution of the optimal control problem under consideration.

We assume the usual definition of a strong local minimizer; we say that the process (x^*, u^*) is a *strong local minimizer* if, for some $\varepsilon > 0$, it minimizes the cost over admissible processes (x, u) such that $|x(t) - x^*(t)| \leq \varepsilon$ for all $t \in [a, b]$.

Throughout this chapter we assume the following *Basic Hypotheses*:

BH1 the functions L and f are $\mathcal{L} \times \mathcal{B}$-measurable,

BH2 the multifunction $U(t)$ has $\mathcal{L} \times \mathcal{B}$-measurable graph,

BH3 the set E is closed and l is locally Lipschitz.

In addition to the basic hypotheses **BH1** – **BH3**, we need to impose two important assumptions **AH1** – **AH2** related to Lipschitz continuity of the functions f and/or L and another assumption **AH3** related to state constraints on the data of our problem (P).

In doing so, let us take a generic function $\phi(t, x, u)$ defined in $[a, b] \times \mathbb{R}^n \times \mathbb{R}^k$ and taking values in $\mathbb{R}^n$ or $\mathbb{R}$.

AH1 There exist constants k_x^ϕ and k_u^ϕ such that for almost every $t \in [a, b]$ and every (x_i, u_i) $(i = 1, 2)$ such that

$$x_i \in \{x : |x - x^*(t)| \leq \varepsilon\}, \qquad u_i \in U(t)$$

we have

$$|\phi(t, x_1, u_1) - \phi(t, x_2, u_2)| \leq k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

AH2 The set valued function $t \to U(t)$ is closed valued and there exists a constant $c > 0$ such that for almost every $t \in [a, b]$ we have

$$|u(t)| \leq c \quad \forall u \in U(t).$$

When **AH1** is imposed on f and/or L, then the Lipschitz constants are denoted by k_x^f, k_u^f, k_x^L and k_u^L. Observe that if $U(t)$ is independent of time, then **AH2** states that the set $U(t)$ is compact.

The assumption **AH2** is strong since we are assuming the controls to be bounded. However this requirement simplifies the proofs of the forthcoming results where limits of sequence of controls need to be taken. Moreover this type of assumption is also satisfied by the problems arising in many real world applications.

On the state constraints h we impose the following assumption:

AH3 There exists a constant $k_h > 0$ such that the function $x \to h(t, x)$ is Lipschitz of rank k_h for all $t \in [a, b]$. Furthermore for all x, the function $t \to h(t, x)$ is continuous except on a finite number of points in $]a, b[$ and at any point t_k the following holds:

$$\lim_{s \to t_k^-} h(s, x) \text{ exists}, \ \lim_{s \to t_k^-} h(s, x) \leq h(t_k, x), \ \text{and} \ \lim_{s \to t_k^+} h(s, x) = h(t_k, x).$$

In **AH3** we do not assume upper semi-continuity of $t \to h(t,x)$ for technical reasons. In this respect we refer to see [24]. However we assume something weaker than continuity. Indeed, **AH3** is an adaptation of an analogous hypothesis in [26].

Next we explore some consequences of our hypotheses that will be relevant in the forth coming analysis.

Let $t \in [a,b]$ such that **AH1** holds when applied to f. Take any $x \in x^*(t) + \varepsilon\mathbb{B}$ and any $u \in U(t)$. Then:

$$\begin{aligned}
|f(t,x,u)| &= |f(t,x,u) - f(t,x^*(t),u^*(t)) + f(t,x^*(t),u^*(t))| \\
&\le |f(t,x,u) - f(t,x^*(t),u^*(t))| + |f(t,x^*(t),u^*(t))| \\
&\le k_x^f|x - x^*(t)| + k_u^f|u - u^*(t)| + |\dot{x}^*(t)| \\
&\le k_x^f\varepsilon + 2k_u^f c + |\dot{x}^*(t)|
\end{aligned}$$

Set $K_f(t) = k_x^f\varepsilon + 2k_u^f c + |\dot{x}^*(t)|$. Then $K_f \in L^1$. Thus

$$|f(t,x,u)| \le K_f(t) \quad \text{a.e. } t \in [a,b] \tag{5.1.1}$$

Similar relation holds for L indeed. Since (x^*, u^*) solves problem (P), the function $t \to L(t,x^*(t),u^*(t))$ is L^1. Thus there exists $K_L \in L^1$ such that for almost every $t \in [a,b]$, any $x \in x^*(t) + \varepsilon\mathbb{B}$ and any $u \in U(t)$ we have

$$|L(t,x,u)| \le K_L(t) \quad \text{a.e. } t \in [a,b] \tag{5.1.2}$$

Also

$$f(t,x,U(t)) \text{ and } L(t,x,U(t)) \text{ are compact for all } x \in x^*(t) + \varepsilon\mathbb{B}. \tag{5.1.3}$$

5.2 Auxiliary Results

We start with the well-known Gronwall's inequality in the integral form. Usually this result is stated assuming for K function to be continuous (see [39]). Here K is assumed to be integrable function.

Lemma 5.2.1 *Let x be a real continuous function and K and v be nonnegative integrable* *functions defined in* $[a,b]$*. We suppose that on* $[a,b]$ *the following inequality holds:*

$$|x(t)| \leq v(t) + \int_a^t K(\tau)|x(\tau)|\ d\tau. \tag{5.2.4}$$

Then

$$|x(t)| \leq v(t) + \int_a^t \exp\Big(\int_s^t K(\sigma)\ d\sigma\Big) K(s)v(s)\ ds.$$

See its proof in the Appendix in Lemma A.6.2.

We next state a simplified version, which we say an adaptation of Theorem 3.1 in [19] (see also [18]) essential to our forthcoming analysis. It applies to the following optimal control problem without state constraints.

$$(S)\quad \begin{cases} \text{Minimize } l(x(a),x(b)) + \displaystyle\int_a^b L(t,x(t),u(t))dt \\ \text{subject to} \\ \ \dot{x}(t) = f(t,x(t),u(t)) \text{ a.e. } t \in [a,b] \\ \ u(t) \in U(t) \text{ a.e. } t \in [a,b] \\ \ (x(a),x(b)) \in E. \end{cases}$$

where the data are as stated for problem (P).

Theorem 5.2.2 *Let* (x^*,u^*) *be a strong local minimizer for problem* (S)*. Assume that the basic assumptions are satisfied,* f *and* L *satisfy* **AH1** *and* U *is closed valued. Then there exist* $p \in W^{1,1}([a,b];\mathbb{R}^n)$ *and a scalar* $\lambda_0 \geq 0$ *satisfying*

the nontriviality condition *[NT]:*

$$||p||_\infty + \lambda_0 > 0,$$

the Euler adjoint inclusion *[EI]:*

$$\begin{aligned}(-\dot{p}(t),0) \in \ \ &\partial^C_{x,u}\Big(\langle p(t), f(t,x^*(t),u^*(t))\rangle - \lambda_0 L(t,x^*(t),u^*(t)) \\ &\qquad\qquad -|p(t)|K d_{U(t)}(u^*(t))\Big) \quad a.e.,\end{aligned}$$

the global Weierstrass condition *[W]:*

$$\begin{aligned}&\forall u \in U(t), \quad \langle p(t), f(t,x^*(t),u)\rangle - \lambda_0 L(t,x^*(t),u) \\ &\leq \langle p(t), f(t,x^*(t),u^*(t))\rangle - \lambda_0 L(t,x^*(t),u^*(t)) \quad a.e.,\end{aligned}$$

and the transversality condition *[T]:*

$$(p(a), -p(b)) \in N_E^L(x^*(a), x^*(b)) + \lambda_0 \partial^L l(x^*(a), x^*(b)).$$

Remark 5.2.3 *In our context, K in* **[EI]** *is a constant depending merely on k_x^f and k_x^L and since $|p(t)|K d_{U(t)}(u^*(t)) \subset N_{U(t)}^C(u^*(t))$,* **[EI]** *can be written as*

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C\big(\langle p(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t))\big) - \{0\} \times N_U^C(u^*(t)).$$

We say that Theorem 5.2.2 is an adaptation of Theorem 3.1 in [19] since the later holds under weaker assumptions. In particular, Theorem 3.1 in [19] assumes that (x^*, u^*) is merely a *local minimizer of radius* R and assumption **AH1** is replaced by a weaker condition where the Lipschitz constants are integrable functions. Also the assumption **AH2** is not imposed.

Theorem 5.2.2 differs from the main result in [22] because of the presence of **[W]**. Condition **[EI]** is a main feature of the necessary conditions established in [22]. For **[EI]** to hold it is important to assume that both f and L are Lipschitz with respect to (x, u). The Lipschitz continuity with respect to the control instead of measurability as in the classical case may come as a disadvantage. However this is the feature responsible for the fact that Proposition 4.1 in [22] holds asserting that the necessary conditions are also sufficient for linear-convex problems (as defined in [22]) in the normal form (when $\lambda_0 = 1$).

In [22], the results are proved under the assumption that (x^*, u^*) is a *weak local minimizer* instead of a strong local minimizer. However, as pointed out in [19], we only need to restrict the controls to $U(t) \cap \mathbb{B}_\varepsilon(u^*(t))$ to see that conditions of Theorem 5.2.2 holds for such weak notion of minimizer.

To finish this discussion we point out that the conditions given by the classical nonsmmoth maximum principle (see [15]) are **[NT]**, **[W]**, **[T]** and **[EI]** is replaced by

$$-\dot{p}(t) \in \partial_x^C\big(\langle p(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t))\big). \tag{5.2.5}$$

The special feature of (5.2.5) is that the adjoint inclusion is decoupled from the control variable, more precisely the generalized gradient is being taken solely with respect to x. We refer the reader to [22] for a detailed discussion on (5.2.5) and **[EI]**.

5.3 Nonsmooth Maximum Principle for (P)

We now turn to problem (P) with its full generality. We will derive a nonsmooth maximum principle for this state constrained problem. Firstly the result is established assuming *convexity* of the "velocity set" (see **C** below). Such hypothesis is then removed following an approach introduced in [67] and explored in [25].

5.3.1 The Convex Case

We shall state our main result. Consider the following hypothesis.

C The velocity set

$$\{(v,l)=(f(t,x,u),L(t,x,u)),\ u\in U(t)\}$$

is convex for all $(t,x)\in[a,b]\times\mathbb{R}^n$.

Also we introduce the following subdifferential

$$\bar{\partial}_x h(t,x) := \text{co}\,\{\lim \xi_i \ :\ \xi_i\in\partial_x h(t_i,x_i),(t_i,x_i)\to(t,x)\}. \tag{5.3.6}$$

Proposition 5.3.1 Let (x^*,u^*) be a strong local minimizer for problem (P). Assume that f and L satisfy **AH1**, that **BH1** – **BH3**, **AH2** and **C** hold and h satisfies **AH3**. Then there exist $p\in W^{1,1}([a,b];\mathbb{R}^n)$, $\gamma\in L^1([a,b];\mathbb{R})$, a measure $\mu\in C^{\oplus}([a,b];\mathbb{R})$, and a scalar $\lambda_0\geq 0$ satisfying

(i) $\mu\{[a,b]\}+||p||_\infty+\lambda_0>0$,

(ii) $(-\dot{p}(t),0)\in$
$\partial^C_{x,u}\Big(\langle q(t),f(t,x^*(t),u^*(t))\rangle-\lambda_0 L(t,x^*(t),u^*(t))\Big)-\{0\}\times N^C_{U(t)}(u^*(t))$ a.e.,

(iii) $\forall\ u\in U(t)$,
$\langle q(t),f(t,x^*(t),u)\rangle-\lambda_0 L(t,x^*(t),u)\leq\langle q(t),f(t,x^*(t),u^*(t))\rangle-\lambda_0 L(t,x^*(t),u^*(t))$ a.e.,

(iv) $(p(a),-q(b))\in N^L_E(x^*(a),x^*(b))+\lambda_0\partial^L l(x^*(a),x^*(b))$,

(v) $\gamma(t)\in\bar{\partial}h(t,x^*(t))\quad\mu$-a.e.,

(vi) $\text{supp}\{\mu\}\subset\{t\in[a,b]:h(t,x^*(t))=0\}$,

where

$$q(t) = \begin{cases} p(t) + \int_{[a,t)} \gamma(s)\mu(ds) & t \in [a,b) \\ p(t) + \int_{[a,b]} \gamma(s)\mu(ds) & t = b. \end{cases} \tag{5.3.7}$$

5.3.2 Nonconvex Case

We now replace the subdifferential $\bar{\partial}_x h$ by a *more refined* subdifferential $\partial_x^> h$ defined by

$$\partial_x^> h(t,x) := \text{co}\ \{\xi : \exists\ (t_i, x_i) \xrightarrow{h} (t,x) : h(t_i,x_i) > 0\ \forall i,\ \partial_x h(t_i,x_i) \to \xi\}. \tag{5.3.8}$$

to sharpen the necessary conditions of optimality in the Theorem 5.3.2. The (hybrid) subdifferential $\partial_x^> h$ is more refined in the sense that in the definition of $\partial_x^> h$ we only consider the convergent sequences (t_i, x_i) such that $h(t_i, x_i) > 0$, while in the definition of $\bar{\partial}_x h$ any convergent sequence is considered. Thus the set $\bar{\partial}_x h$ is larger than $\partial_x^> h$, i.e.,

$$\partial_x^> h \subseteq \bar{\partial}_x h.$$

So, we say that the subdifferential $\partial_x^> h$ is more refined than that of $\bar{\partial}_x h$.

Theorem 5.3.2 Let (x^*, u^*) be a strong local minimizer for problem (P). Assume that f and L satisfy **AH1**, h satisfies **AH3** and that **AH2** as well as the basic assumptions **BH1**–**BH3** hold. Then there exist an absolutely continuous function p, an integrable function γ, a non-negative measure $\mu \in C^{\oplus}([a,b];\mathbb{R})$, and a scalar $\lambda_0 \geq 0$ such that conditions (i)–(vi) of Proposition 5.3.1 hold with $\partial_x^> h$ as in (5.3.8) replacing $\bar{\partial}_x h$ and where q is as defined in (6.4.9).

As mentioned before the above theorem adapts easily when we assume (x^*, u^*) to be a weak local minimizer instead of a strong local minimizer. Our final refinement is stated as the following Theorem.

Theorem 5.3.3 Let (x^*, u^*) be merely a local $W^{1,1}$-minimizer for problem (P). Then the conclusions of Theorem 5.3.2 hold.

5.4 Proof of the Main Results

Throughout this thesis, we shall prove all our results by assuming that the integral part of the cost (or *running cost*) is zero, that is, $L \equiv 0$. The case of $L \neq 0$ is treated by a standard and well known technique which we say the *state augmentation* technique. We have discussed this transformation technique in Chapter 2 (see also [19]). We recall that proving our main results is based on the convexity assumption stated in Proposition 5.3.1.

5.4.1 Proof of Proposition 5.3.1

The proof of our main Theorem 5.3.2 consists of several steps. First we prove Proposition 5.3.1 when specialized to the following simpler problem.

$$
(\mathrm{Q}) \quad \begin{cases} \text{Minimize } l(x(b)) \\ \text{subject to} \\ \begin{array}{rcll} \dot{x}(t) & = & f(t, x(t), u(t)) & \text{a.e. } \ t \in [a, b] \\ u(t) & \in & U(t) & \text{a.e. } \ t \in [a, b] \\ h(t, x(t)) & \leq & 0 & \text{for all } \ t \in [a, b] \\ (x(a), x(b)) & \in & \{x_a\} \times E_b. & \end{array} \end{cases}
$$

Problem (Q) is a special case of (P) in which $E = \{x_a\} \times E_b$ and $l(x_a, x_b) = l(x_b)$.

We proceed proving the necessary conditions of optimality for our problem (Q) in several steps.

Step 1: *Penalize state-constraint violation.*

Define a sequence of problems penalizing the state-constraint violation. The sequence of problems of interests is

$$
(Q_i) \quad \begin{cases} \text{Minimize } l(x(b)) + i \displaystyle\int_a^b h^+(t, x(t)) \, dt \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \ (x(a), x(b)) \in \{x_a\} \times E_b, \end{cases}
$$

for $i \in \mathbb{N}$ where $h^+(t, x) := \max\{0, h(t, x)\}$. This differs from (Q) by shifting the state constraint into the objective function.

Following the approach in [66] (see also [24]) let us temporarily assume that penalization is effective, i.e., we assume the following interim hypothesis:

[IH] $\lim\limits_{i \to \infty} \inf\{Q_i\} = \inf\{Q\}$.

Step 2: *Application of Ekeland's theorem.*

Set W to be the set of measurable functions $u : [a, b] \to \mathbb{R}^k, \quad u(t) \in U(t)$ a.e. such that there exists a solution of the differential equation $\dot{x}(t) = f(t, x(t), u(t))$, for almost every $t \in [a, b]$, with $x(t) \in x^*(t) + \varepsilon\mathbb{B}$ for all $t \in [a, b]$ and $x(a) = x_a$ and $x(b) \in E_b$. We provide W with the L^1 metric defined by $\Delta(u, v) :=\| u - v \|_{L_1}$ and set

$$J_i(u) := l(x(b)) + i \int_a^b h^+(t, x(t))\, dt.$$

Then it is easy to show that (W, Δ) is a complete metric space in which the functional $J_i \colon W \to \mathbb{R}$ is continuous (see [12] for example).

We apply Ekeland's theorem to the sequence of problems of the form

$$(O_i) \begin{cases} \text{Minimize} & J_i(u), \\ \text{subject to} & u \in W. \end{cases}$$

Observe that u^* (corresponding to x^*) is an admissible solution for each of these problems and we have $J_i(u^*) = l(x^*(b)) = \inf Q$.

Let $\varepsilon_i = J_i(u^*) - \inf Q_i$. Then $\varepsilon_i \geq 0$ and taking **[IH]** into account, we get $\varepsilon_i \to 0$. Ekeland's variational principle (see [67]) applies to the sequence of problems (O_i). It asserts the existence of $u_i \in W$ such that

$$\| u_i - u^* \|_{L_1} \leq \sqrt{\varepsilon_i} \tag{5.4.9}$$

and u_i minimizes over W, the functional

$$u \mapsto J_i(u) + \sqrt{\varepsilon_i} \| u_i - u^* \|_{L_1} . \tag{5.4.10}$$

Let x_i be the trajectory corresponding to u_i.

Step 3: *Study optimality conditions for the perturbed problem.*

The conclusions of Ekeland's theorem can be restated as: for each $i \in \mathbb{N}$, (x_i, u_i) solves the following optimal control problem:

$$(E_i)\quad \begin{cases} \text{Minimize } l(x(b)) + i\displaystyle\int_a^b h^+(t,x(t))\,dt + \sqrt{\varepsilon_i}\int_a^b |u(t)-u_i(t)|\ dt \\ \text{subject to} \\ \ \dot{x}(t) = f(t,x(t),u(t)) \text{ a.e. } t\in[a,b] \\ \ u(t)\in U(t) \text{ a.e. } t\in[a,b] \\ \ x(t)\in x^*(t)+\varepsilon\mathbb{B} \ \text{ for all } \ t\in[a,b] \\ \ x(a)=x_a. \end{cases}$$

The fact that $\varepsilon_i \to 0$ allows us to prove that $x_i(a) \to x^*(a)$ and u_i converges strongly to u^*. It follows that there exists a subsequence $\{u_i\}$ (we do not relabel) converging to u^* for almost every $t \in [a,b]$. Along the corresponding subsequence of problems (E_i) we deduce that x_i converges uniformly to x^*. Discarding initial terms of this sequence, if necessary, we guarantee that $x_i(t) \in x^*(t) + \frac{\varepsilon}{2}\mathbb{B}$ for almost every $t \in [a,b]$. Thus the sequence (x_i, u_i) solves a variant of (E_i) obtained when the state constraint is absent. From now on (E_i) denotes such sequence of problems.

We apply Theorem 5.2.2 to each (E_i) obtaining the existence of an absolutely continuous function p_i, an integrable function θ_i and a scalar $\lambda_i \geq 0$ such that

$$(p_i(t),\lambda_i) \neq 0 \text{ for all } t, \tag{5.4.11}$$

$$\begin{aligned}(-\dot{p}_i(t),\theta_i(t)) \in\ & \partial^C_{x,u}\left\{\langle p_i(t), f(t,x_i(t),u_i(t))\rangle - i\lambda_i h^+(t,x_i(t))\right.\\ & \left.-\sqrt{\varepsilon_i}\lambda_i\,|u(t)-u_i(t)|\right\},\end{aligned} \tag{5.4.12}$$

$$\theta_i(t) \in N^C_{U(t)}(u_i(t)) \quad \text{for all } t, \tag{5.4.13}$$

$$\begin{gathered} u\in U(t) \Longrightarrow \\ \langle p_i(t), f(t,x_i(t),u)\rangle - \sqrt{\varepsilon_i}\lambda_i\,|u-u_i(t)| \leq \langle p_i(t), f(t,x_i(t),u_i(t))\rangle \text{ a.e.}\end{gathered} \tag{5.4.14}$$

$$-\,p_i(b) \in N^L_{E_b}(x_i(b)) + \lambda_i\partial^L l(x_i(b)) \tag{5.4.15}$$

These conditions have consequences in terms of the original problem (Q). To discuss the these consequences, let us go for a detailed analysis.

We apply Clarke's sum rule [12] to (5.4.12) and take into account the properties of the subdifferentials. We deduce that there exist measurable functions $\xi_i,\ \zeta_i,\ \eta_i,\ e_i$ such that for almost every t in $[a,b]$,

$$(\xi_i(t),\zeta_i(t)) \in \partial^C_{x,u} f(t,x_i(t),u_i(t)), \tag{5.4.16}$$

$$(\eta_i(t),0) \in \partial^C_{x,u} h^+(t,x_i(t)), \tag{5.4.17}$$

$$e_i(t)\ :\ |e_i(t)| \le 1 \text{ and } e_i(t) \in \partial^C_u |u(t)-u_i(t)|, \tag{5.4.18}$$

$$\theta_i(t) \in N^C_{U(t)}(u_i(t)), \tag{5.4.19}$$

such that

$$-\dot p_i(t) = p_i(t)\xi_i(t) - i\lambda_i\eta_i(t), \tag{5.4.20}$$

$$\theta_i(t) = p_i(t)\zeta_i(t) - \sqrt{\varepsilon_i}\lambda_i e_i(t). \tag{5.4.21}$$

For further simplification, let us consider $h_0(t,x)=0$ and $h_1(t,x)=h(t,x)$ so that

$$h^+(t,x) = \max\{h_j(t,x)\ :\ j=0,1\}.$$

Then for each fixed t, Clarke's Max Rule [12] says

$$\partial^C_{x,u} h^+(t,x_i(t)) \quad \subseteq \partial^C_{x,u} \cup_{j=0}^{1} \left\{\partial^C_{x,u} h_j(t,x_i(t))\ :\ h_j(t,x_i(t)) = h^+(t,x_i(t))\right\}.$$

Since $\partial^C_{x,u} h_0 \equiv \{(0,0)\}$, a typical element of the right side has the form $\alpha_i(t)\,(\gamma_i(t),\ 0)$, where $(\gamma_i(t),0) \in \partial^C_{x,u} h_1(t,x_i(t))$ and $\alpha_i(t) \in \Sigma_i(t)$ where

$$\Sigma_i(t) = \left\{\alpha\ \in [0,1],\ \alpha = 0 \text{ if } h_1(t,x_i(t)) < h^+(t,x_i(t))\right\}.$$

Taking these dependencies into account the following expansion of (5.4.20) and (5.4.21) holds:

$$\begin{cases} -\dot p_i(t) = p_i(t)\xi_i(t) - i\lambda_i\alpha_i(t)\gamma_i(t), \\ \quad \theta_i(t) = p_i(t)\zeta_i(t) - \sqrt{\varepsilon_i}\lambda_i e_i(t). \end{cases} \tag{5.4.22}$$

We now introduce the measure $\mu_i \in C^*([a,b];\mathbb{R})$:

$$\int_B d\mu_i(t) = \int_B i\lambda_i\alpha_i(t)dt$$

for every Borel set $B \subset [a,b]$.

Define $\pi_i \in C^*([a,b];\mathbb{R})$ as $d\pi_i(t) = \dot{p}_i(t)dt$. Then, from (5.4.22) we get

$$\begin{cases} -\displaystyle\int_B d\pi_i(t) = & \displaystyle\int_B \big(p_i(t)\xi_i(t)\big)dt - \int_B \gamma_i(t)d\mu_i(t), \\ \displaystyle\int_B \theta_i(t) \;\; = & \displaystyle\int_B \big(p_i(t)\zeta_i(t) - \sqrt{\varepsilon_i}\lambda_i e_i(t)\big)dt, \end{cases} \tag{5.4.23}$$

and

$$p_i(t) = b_i + \int_{[a,t)} d\pi_i(t) \;\text{ for all } t \in (a,b], \tag{5.4.24}$$

for every Borel set B. Here $b_i = p_i(a)$. Taking (5.4.15) into account we have

$$-b_i - \int_{[a,b]} d\pi_i(t) \in N^L_{E_b}(x_i(b)) + \lambda_i \partial^L l(x_i(b)). \tag{5.4.25}$$

Since $\alpha_i(t) \in \Sigma_i(t)$, we have $\mu_i \in C^{\oplus}([a,b];\mathbb{R})$ and this measure has support in

$$\{t \in [a,b] : h(t, x_i(t)) = 0\}.$$

Since, by (5.4.11), b_i and λ_i are not both zero, we may conclude, after rescaling, that

$$|b_i| + |\mu_i| + \lambda_i = 1. \tag{5.4.26}$$

Step 4: *Take limits.*

Since $\varepsilon_i \to 0$, u_i converges strongly to u^* and x_i converges uniformly to x^*. Consequently, there exists a subsequence such that $u_i(t) \to u^*$ a.e. From now on we work with the corresponding subsequences.

Now we also recall that for any Borel set $B \subset [a,b]$[1]

$$\pi_i(B) = \int_B \dot{p}_i(t)\; dt.$$

We know that p_i is an absolutely continuous function and so is $\dot{p}_i \in L^1([a,b];\mathbb{R}^n)$. Then π_i is absolutely continuous with respect to Lebesgue measure m. It follows that there exists an absolutely continuous function such that

$$P_i(t) := \pi_i([a,t[).$$

[1] Observe that $\pi_i(B) = \displaystyle\int_B d\pi_i$.

Now we focus on (5.4.23) and (5.4.24). Then we have

$$-P_i(t) = \int_{[a,t[} (b_i + P_i(s))\xi_i(t)\ ds - \int_{[a,t[} \gamma_i(t) d\mu_i(t). \tag{5.4.27}$$

By the properties of subdifferentials and **AH1** we know that $|\xi_i(t)| \leq K_f$ where $K_f = \max\{k_f^x, k_f^u\}$. Recall also that by **AH3** we have

$$|\gamma_i(t)| \leq k_h \quad \text{ a.e. } t \in [a,b].$$

Set $A_i(t) := \displaystyle\int_{[a,t[} b_i\ dt - \int_{[a,t[} \gamma_i(t) d\mu_i(t)$.

Since μ_i is a (positive) measure by definition, it follows that

$$\Big| \int_{[a,t[} \gamma_i(t) d\mu_i(t) \Big| \leq \int_{[a,t[} k_h d\mu_i(t) = k_h \int_{[a,b]} d\mu_i(t) = k_h |\mu_i|.$$

Taking (5.4.26) into account we deduce that

$$|A_i(t)| \leq |b_i|(t-a) + k_h|\mu_i| \leq (b-a) + k_h. \tag{5.4.28}$$

Set $v_i(t) = |A_i(t)|$. Clearly this function is integrable. From the above and (5.4.27) we have

$$|P_i(t)| \leq v_i(t) + \int_a^t K_f |P_i(t)|\ dt. \tag{5.4.29}$$

Applying Lemma 5.2.1 to (5.4.29) we get

$$|P_i(t)| \leq v_i(t) + K_f \int_a^t e^{K_f(t-s)} v_i(t)\ dt \quad \text{ for all } t \in [a,b].$$

Furthermore, by (5.4.28) we get

$$|P_i(t)| \leq (b-a) + k_h + K_f e^{K_f(a-b)}[(b-a) + k_h].$$

Set $K_1 = (b-a) + k_h + K_f e^{K_f(a-b)}[(b-a)+k_h]$. We deduce from the above that $|\pi_i| \leq K_1$. It follows from (5.4.24) and (5.4.26) that

$$|p_i(t)| \leq K_1 + 1.$$

Since the sequence $\{\pi_i\}$ is uniformly bounded, we deduce from Banach-Alaoglu's Theorem (see [48]) that

$$\pi_i \to \pi \quad \text{weakly}^*$$

for some measure π. Turning again to (5.4.26) we may arrange that $b_i \to b, \quad \lambda_i \to \lambda$ for some $b \in \mathbb{R}^n$, $\lambda \geq 0$. Moreover, since $|\mu_i| \leq 1$ we also have $\mu_i \to \mu$ weakly* for some measure μ. Also $|\mu_i| \to |\mu|$ and

$$|b| + |\lambda| + |\mu| = 1.$$

With the above and appealing to Lemma 4.3 in [66] we can now conclude that there exists some subsequence such that $p_i(t) \to q(t) \quad$ a.e., where q is now a function of bounded variation defined as

$$q(t) := b + \int_{[a,t)} d\pi,$$

and

$$b_i + \int_{[a,t)} d\pi_i \to b + \int_{[a,t)} d\pi.$$

Under the hypotheses we deduce from (5.4.16) that $|(\xi_i(t), \zeta_i(t))| \leq K_f$ a.e. Dunford-Pettis Theorem (see for example [67, Theorem 2.51]) asserts existence of a subsequence converging weakly in the L^1 topology to some function (ξ, ζ) such that $\xi,\ \zeta \in L^1$. Taking into account (5.4.18) we deduce in the same way that $e_i \to e, \quad$ for some $e \in L^1$ where the convergent is understood in the weak L^1 topology. Upper semi-continuity properties of the subdifferentials asserts that (5.4.16)–(5.4.19) hold when we remove the indexes i.

Observe that $\partial_x^C h(t,x) \subset \bar{\partial}_x h(t,x)$ (see (5.3.6) for definition of $\bar{\partial}_x h(t,x)$) and that $\bar{\partial}_x h(t,x)$ is of closed graph for any i. It follows from [66, Lemma 4.3] that there exists a Borel measurable, μ-integrable function γ such that $\gamma(t) \in \bar{\partial}_x h(t, x^*(t)) \quad \mu-$a.e. This is (5.4.36) of the proposition.

We now turn to (5.4.25). The properties of limiting normal cones and limiting subdifferential assert that

$$-b - \int_{[a,b]} d\pi(t) \in N_{E_b}^L(x^*(b)) + \lambda \partial^L l(x^*(b)). \tag{5.4.30}$$

We concentrate on the support of the measure μ. Mimicking the arguments in [24] it is a simple matter to see that $\text{supp}\{\mu\} \subset \{t \in [a,b] :\ h(t, x^*(t)) = 0\}$.

This is conclusion (5.4.37) of the proposition.

We deduce now from (5.4.14) that

$$\langle q(t), f(t, x^*(t), u)\rangle \leq \langle q(t), f(t, x^*(t), u^*(t))\rangle. \tag{5.4.31}$$

Next we focus on (5.4.23). Lemma 4.3 in [66] and our conclusions above assert that

$$\begin{aligned} -q(t) + b &= \int_{[a,t)} \Big(q(s)\xi(s)\Big) ds - \int_{[a,t)} \gamma(s) d\mu(s) \\ \int_{[a,t)} \theta(t) &= \int_{[a,t)} \Big(q(s)\zeta(s)\Big) ds \end{aligned}$$

Define now the function $p(t) := q(t) - \displaystyle\int_{[a,t)} \gamma(s) d\mu(s)$. From the above we recover the conclusions of the proposition. Observe that (5.4.32) follows from $|b| + |\lambda| + |\mu| = 1$.

Combining all the above, we get the required conclusions under the assumption [IH], i.e., we get the existence of an absolutely continuous function $p : [a, b] \to \mathbb{R}^n$, an integrable function $\gamma : [a, b] \to \mathbb{R}^n$, a measure $\mu \in C^{\oplus}([a, b]; \mathbb{R})$ and a scalar $\lambda_0 \geq 0$ such that

$$\mu\{[a, b]\} + ||p||_\infty + \lambda_0 > 0, \tag{5.4.32}$$

$$(-\dot{p}(t), \theta(t)) \in \partial^C_{x,u}\langle q(t), f(t, x^*(t), u^*(t))\rangle - \{0\} \times N^C_{U(t)}(u^*(t)) \quad \text{a.e.} \tag{5.4.33}$$

$$\forall \;\; u \in U(t), \quad \langle q(t), f(t, x^*(t), u)\rangle \leq \langle q(t), f(t, x^*(t), u^*(t))\rangle \quad \text{a.e.} \tag{5.4.34}$$

$$-q(b) \in N^L_{E_b}(x^*(b)) + \lambda_0 \; \partial^L l(x^*(b)), \tag{5.4.35}$$

$$\gamma(t) \in \bar{\partial} h(t, x^*(t)) \quad \mu\text{-a.e.}, \tag{5.4.36}$$

$$\text{supp}\{\mu\} \subset \{t \in [a, b] : h(t, x^*(t)) = 0\}, \tag{5.4.37}$$

where q is as in (5.3.7).

Step 5: *Show that* $\mathbf{C}$ *implies [IH].*

Now we will show that our interim hypothesis $[IH]$ is equivalent to $\mathbf{C}$.

Choose an admissible process (x_i, u_i) for (Q_i) such that for $i \to \infty$, we have

$$l(x_i(b)) + i \int_a^b h^+(t, x_i(t)) \; dt \leq \inf\{Q_i\} + \frac{1}{i}.$$

Define the multifunction

$$F(t,x) = \{f(t,x,u):\ u \in U(t)\}\,.$$

The basic hypotheses, **AH1**–**AH3** and **C** assert that this multifunction is $\mathcal{L} \times \mathcal{B}$ measurable, nonempty, closed, convex and integrably bounded (see [12] for definitions) on the set

$$\Omega := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n :\ t \in [a,b],\ x \in x^*(t) + \varepsilon\mathbb{B}\}\,.$$

For each $i \in \mathbb{N}$, the set of admissible processes for (Q_i) is nonempty since (x^*, u^*) is admissible for each (Q_i). We have

$$\inf\ Q_i\ \leq \inf\ Q\ . \tag{5.4.38}$$

In view of **C** and Proposition 3.2.3 in [12] (Q_i) has a solution. Let us denote such solution as (x_i^*, u_i^*) and $\bar{J}_i$ the corresponding cost. We have $\bar{J}_i = \inf\ Q_i$.

Now we suppose that there exists an admissible solution (x_i, u_i) of (Q_i) such that

$$l(x_i(b)) + i\int_a^b h^+(t, x_i(t))dt \leq \inf\ Q_i + \frac{1}{i}. \tag{5.4.39}$$

Appealing to Theorem 2.5.3 in [67] we deduce the existence of a subsequence (we do not relabel) such that

$$x_i \to x \quad \text{uniformly} \quad \text{ and } \dot{x}_i \to \dot{x} \quad \text{weakly in } L^1$$

for some absolutely continuous function x with $x(a) = x_a$ and $\dot{x}(t) \in F(t, x(t))$ for almost every $t \in [a,b]$. Appropriate measurable selection theorems guarantee the existence of some measurable function u such that $u(t) \in U(t)$ and $\dot{x}(t) = f(t, x(t), u(t))$. Since $x_i(b)$ is bounded, inequalities (5.4.38) and (5.4.39) assert that $\int_a^b h^+(t, x_i(t))dt$ is bounded. Hence

$$\int_a^b h^+(t, x(t))dt = 0.$$

We claim that $h(t, x(t)) \leq 0$ for all $t \in [a,b]$. To see this suppose that there exists some $\bar{t} \in [a,b]$ such that

$$h(\bar{t}, x(\bar{t})) > 0.$$

Then by **AH3**, there exists some $\delta > 0$ such that for all $t \in (\bar{t}, \bar{t}+\delta)$ (or, if $\bar{t} = b$, $(\bar{t}-\delta, \bar{t}\,]$)

we have $h(t,x(t)) > 0$. Thus $\int_{\bar{t}}^{\bar{t}+\delta} h^+(t,x(t))dt \geq 0$ contradicting $\int_a^b h^+(t,x(t))dt = 0$ and proving our claim.

We conclude that (x,u) is an admissible process for (Q) and

$$l(x(b)) \geq l(x^*(b)).$$

Moreover

$$\begin{aligned} \inf Q \leq l(x(b)) &\leq \lim \left(l(x_i(b)) + \int_a^b h^+(t,x_i(t))dt \right) \\ &\leq \lim \left(\inf Q_i + \frac{1}{i} \right) = \liminf Q_i. \end{aligned} \tag{5.4.40}$$

But (5.4.38) asserts that $\liminf Q_i \leq \inf Q$. Thus $\liminf Q_i = \inf Q$, i.e., **[IH]** is guaranteed by the other hypotheses of Proposition 5.3.1.

Step 6: *Additional stages*

The proof of Proposition 5.3.1 still requires some additional works. The remaining of the proof comprises three stages. We first extend Proposition 5.3.1 to problems where $x(a) \in E_a$, and E_a is a closed set. This will be done following the lines at the end of the proof of Theorem 3.1 in [66]. Thus we obtain necessary conditions when $(x(a),x(b)) \in E_a \times E_b$. Next we consider the case when the cost is $l = l(x(a),x(b))$. This will be done using a technique of [27]. And finally, following the approach in section 6 in [27], we derive necessary conditions when $(x(a),x(b)) \in E$ and E is a closed set. Now we start the details of the proof.

Step 6.a: *$x(a) \in E_a$, and E_a is a closed set.*

We will prove this step by considering a reformulated problem in which we take two control variables (u,w), two state variables (x,z) and the underlying time interval is $[a-1,b]$. First let us define

$$\tilde{U}(t) = \begin{cases} c\mathbb{B} \text{ if } \ t \in [a-1,a) \\ U(t) \text{ if } \ t \in [a,b] \end{cases} \tag{5.4.41}$$

where c is the constant in **AH2** concerning $U(t)$ and $\mathbb{B}$ is the closed ball centered at 0 with radius 1.

Let $W = E_a$ and define

$$\tilde{h}(t,x) = \begin{cases} r \text{ if } \; t \in [a-1,a) \\ h(t,x) \text{ if } \; t \in [a,b] \end{cases} \tag{5.4.42}$$

where $r < 0$ is a fixed lower bound on values of h[2]. Now we consider the problem

$$(N) \quad \begin{cases} \text{Minimize } l(x(b)) \\ \text{subject to} \\ \dot{z}(t) = w(t) \text{ if } t \in [a-1,a), \quad \dot{z}(t) = 0 \text{ if } t \in [a,b] \\ \dot{x}(t) = w(t) \text{ if } t \in [a-1,a), \quad \dot{x}(t) = f(t,x(t),u(t)) \text{ if } t \in [a,b] \\ w(t) \in W \text{ a.e. } t \in [a-1,b] \\ u(t) \in \tilde{U}(t) \text{ a.e. } t \in [a-1,b] \\ \tilde{h}(t,x) \leq 0 \text{ for all } t \in [a-1,b] \\ x(a-1) = 0 \\ z(a-1) = 0 \\ (z(b),x(b)) \in E_a \times E_b. \end{cases}$$

Now we have

$$z(t) = 0 + \int_{a-1}^{t} w(s)ds = x(t) \;\text{ for } t < a$$

which implies

$$z(a) = \int_{a-1}^{a} w(s)ds = x(a).$$

Also we have

$$z(t) = z(a) + \int_{a}^{t} 0 \; dt = z(a) \;\text{ for } t > a.$$

But $z(b) = z(a)$ and $z(b) \in E_a$. Consequently $z(a) \in E_a$. Thus for any admissible process (x,z,u,w) of (N) we have

- $z(a) = x(a)$,
- $z(a) = z(b)$ because $\dot{z}(t) = 0$ for $t \in [a,b]$,
- $z(b) \in E_a$ which implies $x(a) \in E_a$,

[2] r can be chosen to be $r = \min\{-1, \min\{h(t,x) : t \in [a,b], \; x \in x^*(t) + \varepsilon\mathbb{B}\}\}$

- $x(t) = x(a) + \displaystyle\int_a^t f(s, x(s), u(s))\ ds$ for $t > a$,
- $x(b) \in E_b$,
- $x(t) = 0 + \displaystyle\int_{a-1}^t w(s)\ ds$ for $t < a$ and $\tilde{h}(t, x(t)) = r < 0$, since r is negative.

Now we choose $(\tilde{x}, \tilde{z}, \tilde{u}, \tilde{w})$ as

$$\tilde{x}(t) = \begin{cases} x^*(a)(t - (a-1)) & \text{for } \ t < a, \\ x^*(t) & \text{for } \ t \geq a, \end{cases}$$

$$\tilde{z}(t) = \begin{cases} x^*(a)(t - (a-1)) & \text{for } \ t < a, \\ x^*(a) & \text{for } \ t \geq a, \end{cases}$$

$$\tilde{u}(t) = \begin{cases} 0 \text{ for } \ t < a, \\ u^*(t) \qquad \text{for } \ t \geq a, \end{cases}$$

$$\tilde{w}(t) = \begin{cases} x^*(a)(\in E_a) & \text{for } \ t < a, \\ x^*(a) & \text{for } \ t \geq a. \end{cases}$$

Then $(\tilde{x}, \tilde{z}, \tilde{u}, \tilde{w})$ is admissible for our new problem (N) with cost $l(x(b))$. Suppose we have another (x, z, u, w) admissible for the problem (N) with the cost $l(x(b)) < l(x^*(b))$. Then since $z(a) = x(a) = z(b) \in E_a$, we have $x(a) \in E_a$ and also $x(b) \in E_b$, $h(t, x(t)) \leq 0$ for $t \geq a$. Moreover, $\dot{x}(t) = f(t, x(t), u(t))$ for $t \geq a$.

Now we take

$$\begin{cases} \hat{x}(t) = x(t) \text{ for } t \geq a \\ \hat{u}(t) = u(t) \text{ for } t \geq a \\ h(t, \hat{x}(t)) \leq 0 \text{ for } t \geq a \\ \dot{\hat{x}}(t) = f(t, \hat{x}(t), \hat{u}(t)) \text{ for } t \geq a \\ \hat{u}(t) \in U(t) \ \text{ a.e.} \\ (\hat{x}(a), \hat{x}(b)) \in E_a \times E_b. \end{cases}$$

So $(\hat{x}, \hat{u})$ is a solution to our initial problem (Q) with cost

$$l(\hat{x}(b)) < l(x^*(b)).$$

which is a contradiction.

Thus $(\tilde{x}, \tilde{z}, \tilde{u}, \tilde{w})$ solves (N). It is a simple matter to see that Proposition 5.3.1 applies. Applying it we get the required conclusions.

Step 6.b: *Consider the case when the cost is* $l = l(x(a), x(b))$.

Consider the reformulated problem

$$(M)\quad \begin{cases} \text{Minimize } l(z(b), x(b)) \\ \text{subject to} \\ \dot{z}(t) = 0 \text{ if } t \in [a-1, a), \quad \dot{z}(t) = 0 \text{ if } t \in [a, b] \\ \dot{x}(t) = 0 \text{ if } t \in [a-1, a), \quad \dot{x}(t) = f(t, x(t), u(t)) \text{ if } t \in [a, b] \\ u(t) \in \tilde{U}(t) \text{ a.e. } t \in [a-1, b] \\ \tilde{h}(t, x) \leq 0 \text{ for all } t \in [a-1, b] \\ x(a-1) = 0 \\ z(a-1) = 0 \\ (z(a-1), x(a-1)) \in D \\ (z(b), x(b)) \in E_a \times E_b. \end{cases}$$

where $\tilde{U}(t)$ and $\tilde{h}(t, x)$ are defined in (5.4.41) and (5.4.42) and $D = \{(z, w) \in \mathbb{R}^n \times \mathbb{R}^n : z = w\}$. Then clearly the process $(\tilde{x}, \tilde{z}, \tilde{u})$ where

$$\tilde{x}(t) = \begin{cases} x^*(a) & \text{for } \ t \in [a-1, a), \\ x^*(t) & \text{for } \ t \in [a, b], \end{cases}$$

$$\tilde{z}(t) = \begin{cases} x^*(a) & \text{for } \ t \in [a-1, a), \\ x^*(a) & \text{for } \ t \in [a, b], \end{cases}$$

$$\tilde{u}(t) = \begin{cases} c\mathbb{B} & \text{for } \ t \in [a-1, a), \\ u^*(t) & \text{for } \ t \in [a, b], \end{cases}$$

solves the problem (M). Then mimicking with the techniques of [27], it is an easy task to see that the data of (M) satisfies all the basic hypotheses along with the assumptions

AH1–**AH3** and **C**, under which the Proposition 5.3.1 has been shown to apply. Since

$$N_D(x^*(a), x^*(a)) = \{(\theta, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \theta = \eta\},$$

the conclusions of Proposition 5.3.1 hold for problem (P) when

$$l = l(x(a), x(b)), \quad (x(a), x(b)) \in E_a \times E_b.$$

Step 6.c: *Finally consider* $(x(a), x(b)) \in E$

A standard state-augmentation trick converts problem (P) as stated into a problem with separated endpoint constraints: introduce an additional state $y \in \mathbb{R}^n$ with dynamics $\dot{y}(t) = 0$ and impose

$$(x(a), y(a)) \in E, \quad (x(b), y(b)) \in \{(x, y) \in \mathbb{R}^{2n} : x = y\}.$$

The results already obtained apply to the augmented problem, and the stated result for (P) is easily extracted from them.

This completes the proof. ■

5.4.2 Proof of Theorem 5.3.2

We now proceed to prove our main Theorem 5.3.2. We again assume that $L \equiv 0$. We recall that under our hypotheses, (5.1.1) holds and that the set $f(t, x, U(t))$ is compact. Here we follow the approach of [67] and [25].

We first focus on the following 'minimax' optimal control problem where the state constraint functional $\max\limits_{t\in[a,b]} h(t, x(t))$ appears in the cost.

$$(\tilde{R}) \begin{cases} \text{Minimize } \tilde{l}(x(a), x(b), \max\limits_{t\in[a,b]} h(t, x(t))) \\ \text{over } x \in W^{1,1} \text{ and measurable } u \text{ satisfying} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [a, b] \\ \ u(t) \in U(t) \text{ a.e. } t \in [a, b] \\ \ (x(a), x(b)) \in E_a \times \mathbb{R}^n. \end{cases}$$

where $\tilde{l} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $E_a \subset \mathbb{R}^n$ is a given closed set. We observe that $(\tilde{R})$ is the optimal control problem with free endpoint constraints.

We impose here the following additional assumption **AH4**.

AH4 The integrable function $\tilde{l}$ is Lipschitz continuous on a neighbourhood of

$$(x^*(a), x^*(b), \max_{t\in[a,b]} h(t, x^*(t)))$$

and $\tilde{l}$ is monotone in the z variable, in the sense that $z' \geq z$ implies $\tilde{l}(y, x, z') \geq \tilde{l}(y, x, z)$, for all $(y, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

The following proposition is a straightforward adaptation of Proposition 9.5.4 of [67].

Proposition 5.4.1 Let (x^*, u^*) be a strong local minimizer for problem $(\tilde{R})$. Assume the basic hypotheses, **AH1**, **AH2**, **AH3** and the **AH4** hold. Then there exist an absolutely continuous function $p : [a, b] \to \mathbb{R}^n$, an integrable function $\gamma : [a, b] \to \mathbb{R}^n$ and a non-negative measure $\mu \in C^{\oplus}([a, b]; \mathbb{R})$ such that

$$(-\dot{p}(t), 0) \in \partial^C_{x,u}\langle q(t), f(t, x^*(t), u^*(t))\rangle - \{0\} \times N^C_{U(t)}(u^*(t)) \quad \text{a.e.,} \tag{5.4.43}$$

$$(p(a), -q(b), \int_{[a,b)} \mu(ds)) \in \tag{5.4.44}$$

$$N^L_{E_a}(x^*(a)) \times \{0, 0\} + \partial^L \tilde{l}(x^*(a), x^*(b), \max\nolimits_{t\in[a,b]} h(t, x^*(t))),$$

$$\gamma(t) \in \bar{\partial} h(t, x^*(t)) \quad \mu\text{-a.e.,} \tag{5.4.45}$$

$$\forall \;\; u \in U(t), \;\; \langle q(t), f(t, x^*(t), u)\rangle \leq \langle q(t), f(t, x^*(t), u^*(t))\rangle \;\; \text{a.e.,} \tag{5.4.46}$$

$$\text{supp}\{\mu\} \subset \left\{t \in [a, b] : h(t, x^*(t)) = \max\nolimits_{s\in[a,b]} h(s, x^*(s))\right\}, \tag{5.4.47}$$

where q is defined as in (5.3.7).

Proof of Proposition 5.4.1:

We will prove Proposition 5.4.1 mimicking the arguments of the proof of Proposition 9.5.4 in [67].

Since (x^*, u^*) is a *local minimium* for the problem $(\tilde{R})$, there exists a parameter $\varepsilon > 0$ such that

$$\tilde{l}(x^*(a), x^*(b), \max_{t\in[a,b]} h(t, x^*(t))) \leq \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t)))$$

over all admissible processes (x, u) satisfying the constraints of $(\tilde{R})$ and also

$$\|x(t) - x^*(t)\|_\infty \leq \varepsilon.$$

Define the set

$$Q' := \{x \in W^{1,1} : x(a) \in E_a, \quad \dot{x}(t) \in f(t, x(t), U(t))\}.$$

The Generalized Filippov Selection Theorem (see for example [67, Theorem 2.3.13]) guarantees that x^* is a strong local minimizer for the problem

$$\begin{cases} \text{Minimize } \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t))) \\ \text{over arcs } x \in Q' \text{ satisfying } \; \|x - x^*\|_\infty \le \varepsilon. \end{cases}$$

Applying to the Relaxation Theorem (see for example [67, Theorem 2.7.2]), any arc x in the set

$$Q'_r := \{x \in W^{1,1} : x(a) \in E_a, \quad \dot{x}(t) \in \text{co } f(t, x(t), U(t))\}$$

satisfying $\|x - x^*\|_\infty \le \varepsilon$ can be approximated by an arc y in Q' with $\|y - x^*\|_\infty \le \varepsilon$. By **AH4**, the mapping

$$x \to \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t)))$$

is continuous on a neighbourhood of x^*. Thus x^* is a minimizer of

$$\begin{cases} \text{Minimize } \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t))) \\ \text{over } x \in Q'_r \text{ satisfying } \; \|x - x^*\|_\infty \le \varepsilon. \end{cases}$$

Carathéodory's Theorem asserts that there exist $(n+1)$ elements of $f(t, x(t), U(t))$ of the form $f(t, x(t), u_i(t))$, where $u_i \in U(t)$ $i = 0, 1, \ldots, n$, such that

$$y(t) = \lambda_0(t) f(t, x(t), u_0(t)) + \cdots + \lambda_n(t) f(t, x(t), u_n(t))$$

where $\lambda_i \ge 0$, $i = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} \lambda_i = 1$.

In light of the above, the last problem takes now the form

$$\begin{cases} \text{Minimize } \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t))) \\ \text{s.t.} \\ \dot{x}(t) = \Sigma_{i=0}^{n} \lambda_i(t) f(t, x(t), u_i(t)), \quad \text{a.e.,} \\ (\lambda_0(t), \ldots, \lambda_n(t)) \in \Lambda, \\ (u_0(t), \ldots, u_n(t)) \in U(t) \times U(t) \times \ldots \times U(t), \\ x(a) \in E_a \end{cases}$$

where

$$\Lambda = \{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} : \lambda_i \geq 0, \quad i = 0, 1, \ldots, n \ \text{ and } \ \sum_{i=0}^{n} \lambda_i = 1\}.$$

Let us now introduce $z^* := \max_{t \in [a,b]} h(t, x(t))$. So $\dot{z}(t) = 0$ because $\max_{t \in [a,b]} h(t, x(t))$ is a constant. Also by definition of z we should have

$$h(t, x(t)) \leq z(t)$$

which implies that

$$h(t, x(t)) - z(t) \leq 0.$$

Define a new *state variable* y such that $\dot{y}(t) = 0$. We next impose that

$$(x(a), x(b), z(a), y(a)) \in \text{epi}\{\tilde{l}(x(a), x(b), z(a)) + \Psi_{E_a \times \mathbb{R}^n \times \mathbb{R}}\},$$

where Ψ_A represents the indicator function of a set A. This guarantees that

$$x(a) \in E_a, \ \text{ and } y(a) = y(b) \geq \tilde{l}(x(a), x(b), z(a)).$$

Let us put all together and rewrite our problem in the following form

$$(O) \begin{cases} \text{Minimize } y(b) \\ \text{over } x \in W^{1,1}, \ y \in W^{1,1}, \ z \in W^{1,1} \\ \text{and measurable functions } u_0, \ldots, u_n, \lambda_0, \ldots, \lambda_n \text{ satisfying} \\ \dot{x}(t) = \Sigma_i \lambda_i(t) f(t, x(t), u_i(t)), \ \dot{y}(t) = 0, \ \dot{z}(t) = 0 \ \text{ a.e.,} \\ (\lambda_0(t), \ldots, \lambda_n(t)) \in \Lambda, \ u_i(t) \in U(t), \ i = 0, \ldots, n \ \text{ a.e.,} \\ h(t, x(t)) - z(t) \leq 0 \text{ for all } t \in [a, b], \\ (x(a), x(b), z(a), y(a)) \in \text{epi}\{\tilde{l} + \Psi_{E_a \times \mathbb{R}^n \times \mathbb{R}}\} \end{cases}$$

where

$$\Lambda := \{\lambda'_0, \ldots, \lambda'_n : \lambda'_i \geq 0 \text{ for } i = 0, \ldots, n \text{ and } \Sigma_i \lambda'_i = 1\},$$

and $(\lambda_0, \ldots, \lambda_n)$, $(u_0, \ldots, u_n)$ are regarded as control variables.

Now let us take $(x^*, \ u^*, \ z^* = \max h(t, x^*(t)), \ y^*(b) = \tilde{l}(x^*(a), x^*(b), z^*(a)))$. Then

$$\{x^*, y^* \equiv \tilde{l}(x^*(a), x^*(b), z^*), z^*, (u_0^*, \ldots, u_n^*) \equiv (u^*, \ldots, u^*), (\lambda_0, \lambda_1, \ldots, \lambda_n) \equiv (1, 0, \ldots, 0)\}$$

is a minimizer for the problem (O).

But this is a problem to which Proposition 5.3.1 is applicable (because the velocity set is convex). We deduce the existence of absolutely continuous functions $p_1\colon [a,b] \to \mathbb{R}^n$, $p_2\colon [a,b] \to \mathbb{R}$, $p_3\colon [a,b] \to \mathbb{R}$, integrable functions $\xi_j : [a,b] \to \mathbb{R}^m$, $j = 0,1,\dots,n$, $\eta : [a,b] \to \mathbb{R}^{n+1}$ and $\gamma : [a,b] \to \mathbb{R}^n$, a non-negative measure $\mu \in C^{\oplus}([a,b];\mathbb{R})$ and a scalar $\lambda_0 \geq 0$ such that

$$\mu\{[a,b]\} + ||(p_1,p_2,p_3)||_\infty + \lambda_0 > 0, \tag{5.4.48}$$

$$\begin{aligned}&(-\dot{p_1}(t), -\dot{p_2}(t), -\dot{p_3}(t), \xi_0(t), \dots, \xi_n(t), \eta(t)) \in \\ &\quad \partial^C_{x,y,z,u}\langle q_i(t), \big(\Sigma_i \lambda_i(t) f(t, x^*(t), u_i^*(t)), 0, 0\big)\rangle \ \text{ a.e.,}\end{aligned} \tag{5.4.49}$$

$$\xi_i(t) \in N^C_{U(t)}(u^*(t)) \ \text{ a.e., } \ i \in \{0,1,\dots,n\} \tag{5.4.50}$$

$$\eta(t) \in N^C_{\Lambda}(1,0,\dots,0) \ \text{ a.e.,} \tag{5.4.51}$$

$$\forall \ \ u \in U(t), \ \ \langle q(t), f(t,x^*(t),u)\rangle \leq \langle q(t), f(t,x^*(t),u^*(t))\rangle \ \text{ a.e.,} \tag{5.4.52}$$

$$(p_1(a), -q_1(b), p_2(a), p_3(a)) \in \partial \Psi_{\text{epi}\{\tilde{l}+\Psi_{E_a\times\mathbb{R}^n\times\mathbb{R}}\}}(x^*(a), x^*(b), y^*, z^*), \tag{5.4.53}$$

$$-p_2(b) = \lambda_0, \tag{5.4.54}$$

$$-q_3(b) = 0, \tag{5.4.55}$$

$$(\gamma_1(t), \gamma_3(t)) \in \bar{\partial}_{x,z}\big(h(t,x^*(t)) - z^*(t)\big) \quad \mu\text{-a.e.,} \tag{5.4.56}$$

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h\,(t, x^*(t)) = z^*\}\,, \tag{5.4.57}$$

where

$$q_1(t) = \begin{cases} p_1(t) + \int_{[a,t)} \gamma_1(s)\mu(ds) & t \in [a,b) \\ p_1(t) + \int_{[a,b]} \gamma_1(s)\mu(ds) & t = b. \end{cases} \tag{5.4.58}$$

and

$$q_3(t) = \begin{cases} p_3(t) + \int_{[a,t)} \gamma_3(s)\mu(ds) & t \in [a,b) \\ p_3(t) + \int_{[a,b]} \gamma_3(s)\mu(ds) & t = b. \end{cases} \tag{5.4.59}$$

In our case, $\langle (p_1,p_2,p_3), \big(\Sigma_i \lambda_i(t) f(t,x^*(t),u_i^*(t)), 0, 0\big)\rangle$ is independent of y and z which implies,

$$\dot{p_2} = 0 \text{ and } \dot{p_3} = 0.$$

Thus we only need to calculate the Clarke's subdifferential of $\langle p_1, (\Sigma_i \lambda_i f(t,x,u_i))\rangle$ at the point $\big(x^*(t), q_1(t), (u^*(t),\dots,u^*(t)), (1,0,\dots,0)\big)$ which, with the help of sum and product rules of (see Theorems 5.4.1 and 5.4.2 in [67]) yields

$$\partial^C_{x,p_1,(u_0,\ldots,u_n),(\lambda_0,\ldots,\lambda_n)}\langle\lambda_i q_1,(f(t,x,u_i)\rangle.$$

Now for each $i\in\{0,\ldots,n\}$ we deduce that

$$\begin{gathered}\partial^C_{x,p_1,(u_0,\ldots,u_n),(\lambda_0,\ldots,\lambda_n)}\langle\lambda_i q_1,(f(t,x,u_i)\rangle\subset\\ \{(\lambda_i\mu_i,\lambda_i f(t,x,u_i),(0,\ldots,\overbrace{\lambda_i\nu_i}^{\text{ith term}},\ldots,0),(0,\ldots,\overbrace{q_1.f(t,x,u_i)}^{\text{ith term}},\ldots,0)):\\ (\mu_i,\nu_i)\in\partial^C_{x,u}\langle q_1,(f(t,x,u_i)\rangle\}.\end{gathered}\tag{5.4.60}$$

Therefore, the equation (5.4.49) takes the form,

$$\begin{aligned}&(-\dot{p_1}(t),\xi_0(t))\in\partial^C_{x,u}\langle q_1(t),f(t,x^*(t),u^*(t))\rangle,\\&\xi_1(t)\equiv\ldots\equiv\xi_n(t)\equiv 0,\\&\eta(t)=(q_1(t).f(t,x^*(t),u^*(t)),\ldots,(q_1(t).f(t,x^*(t),u^*(t))).\end{aligned}\tag{5.4.61}$$

Also the multiplier η defined above satisfies equation (5.4.51). From (5.4.53) to (5.4.55) we get

$$(p_1(a),-q_1(b),\int_{[a,b]}\mu(ds))\in\lambda_0\partial^L\tilde{l}(x^*(a),x^*(b),z^*)+N^L_{E_a}(x^*(a))\times\{(0,0)\},\tag{5.4.62}$$

Using the above relations and taking into account (5.4.48) we can write $(p_1,\mu,\lambda_0)\neq 0$. On the other hand, we must have $\lambda_0>0$, because if $\lambda_0=0$, then from (5.4.62) we get $\mu=0$ and $p_1(b)=0$ and we get $p_1\equiv 0$, which is impossible. So, rescaling the multipliers we get $\lambda_0=1$. This proves Proposition 5.4.1. ■

Now we prove Theorem 5.3.2.

Consider

$$\begin{aligned}V:=\{(x,u,e):(x,u)&\text{ satisfies }\dot{x}(t)=f(t,x(t),u(t)),\\ u(t)\in U(t)\text{ a.e., }&e\in\mathbb{R}^n,(x(a),e)\in E\text{ and }\|x-x^*\|_\infty\leq\varepsilon\}\end{aligned}\tag{5.4.63}$$

and let $d_V:V\times V\to\mathbb{R}$ be a function defined by

$$d_V((x,u,e),(x',u',e'))=|x(a)-x'(a)|+|e-e'|+\int_a^b|u(t)-u'(t)|dt\tag{5.4.64}$$

For all i, we choose $\varepsilon_i \downarrow 0$ and define the function

$$\tilde{l}_i(x, y, x', y', z) := \max\{l(x, y) - l(x^*(a), x^*(b)) + \varepsilon_i^2, z, |x' - y'|\}$$

Consider the optimization problem

$$\text{Minimize } \{\tilde{l}_i(x(a), e, x(b), e, \max_{t\in[a,b]} h(t, x(t))) : (x, u, e) \in V\}$$

The function d_V defines a metric on the set V and (V, d_V) is a metric space with the following properties:

(i) (V, d_V) is a complete metric space on V,

(ii) If $(x_i, u_i, e_i) \to (x, u, e)$ in the metric space (V, d_V), then $\|x_i - x\|_\infty \to 0$,

(iii) The function $(x, u, e) \to \tilde{l}_i(x(a), e, x(b), e, \max_{t\in[a,b]} h(t, x(t)))$ is continuous on (V, d_V).

We observe that

$$\tilde{l}_i(x^*(a), x^*(b), x^*(b), x^*(b), \max_{t\in[a,b]} h(t, x^*(t))) = \varepsilon_i^2.$$

Since $\tilde{l}_i$ is non-negative, it follows that $(x^*, u^*, x^*(b))$ is an ε_i^2-minimizer for the above minimization problem. According to Ekeland's Variational Principle there exists a sequence $\{(x_i, u_i, e_i)\}$ in V such that, for each i, we have

$$\begin{aligned} &\tilde{l}_i(x_i(a), e_i, x_i(b), e_i, \max\nolimits_{t\in[a,b]} h(t, x_i(t))) \leq \\ &\tilde{l}_i(x(a), e, x(b), e, \max\nolimits_{t\in[a,b]} h(t, x(t))) + \varepsilon_i d_V((x, u, e), (x_i, u_i, e_i)) \end{aligned} \tag{5.4.65}$$

for all $(x, u, e) \in V$ and

$$d_V((x_i, u_i, e_i), (x^*, u^*, x^*(b))) \leq \varepsilon_i. \tag{5.4.66}$$

Condition (5.4.66) implies that $e_i \to x^*(b)$ and $u_i \to u^*$ in the L^1 norm. By using the subsequence extraction, we say that $u_i \to u^*$ a.e. and $x_i \to x^*$ uniformly.

Now we define the arc $y_i \equiv e_i$. Accordingly we get $y_i \to x^*(b)$ uniformly. From the minimization property (5.4.65), we say that $(x_i, y_i, w_i \equiv 0, u_i)$ is a strong local minimizer for the optimal control problem

$$(\tilde{R}_i)\quad \begin{cases} \text{Minimize } \tilde{l}_i(x(a), y(a), x(b), y(b), \max_{t\in[a,b]} h(t, x(t))) \\ \quad + \varepsilon_i[|x(a) - x_i(a)| + |y(a) - y_i(a)| + w(b)] \\ \text{over } x, y, w \in W^{1,1} \text{ and measurable functions } u \text{ satisfying} \\ \quad \dot{x}(t) = f(t, x(t), u(t)), \dot{y}(t) = 0, \dot{w}(t) = |u(t) - u_i(t)| \text{ a.e.,} \\ \quad u(t) \in U(t) \text{ a.e.,} \\ \quad (x(a), y(a), w(a)) \in E \times \{0\}. \end{cases}$$

We observe that the cost function of $(\tilde{R}_i)$ satisfies the assumption **AH4** of Proposition 5.4.1. Thus this is an example of optimal control problem where the special case of maximum principle of Proposition 5.4.1 applies.

We then deduce the existence of absolutely continuous functions $p_i \in W^{1,1}$, $d_i \in \mathbb{R}^n$, $r_i \in \mathbb{R}$, integrable functions ξ_i and γ_i, a non-negative measure $\mu_i \in C^{\oplus}([a, b]; \mathbb{R})$ satisfying

$$\begin{gathered} (-\dot{p}_i(t), -\dot{d}_i(t), -\dot{r}_i(t), \dot{x}_i(t), \dot{y}_i(t), 0, \xi_i(t)) \in \\ \partial^C\left(\langle q_i(t), f(t, x_i(t), u_i(t))\rangle + r_i|u(t) - u_i(t)|\right) \quad \text{a.e.,} \\ \xi_i(t) \in N^C_{U(t)}(u_i(t)) \quad \text{a.e. } t \in [a, b], \end{gathered} \tag{5.4.67}$$

$$\forall \quad u_i \in U(t), \quad \langle q_i(t), f(t, x_i(t), u)\rangle + r_i|u(t) - u_i(t)| \le \langle q_i(t), f(t, x_i(t), u_i(t))\rangle \quad \text{a.e.,}$$

$$\begin{gathered} (p_i(a), d_i(a), r_i(a), -q_i(b), -d_i(b), -r_i(b), \textstyle\int_{[a,b]} \mu_i(dt)) \in \\ N^L_{E\times\{0\}}(x_i(a), y_i(a), w_i(a)) \times \{0, 0, 0, 0\} + \partial^L\{\tilde{l}_i(x, y, x', y', z) \\ +\varepsilon_i[|x(a) - x_i(a)| + |y(a) - y_i(a)| + w_i(b)]\}, \end{gathered} \tag{5.4.68}$$

$$\gamma_i(t) \in \bar{\partial}_x h(t, x_i(t)) \quad \mu\text{-a.e.,}$$

$$\operatorname{supp}\{\mu_i\} \subset \left\{t : h\left(t, x_i(t)\right) = \max_{s\in[a,b]} h(s, x_i(s))\right\}.$$

where $q_i := p_i + \int \gamma_i(s)\mu_i(ds)$ in the above relations. We identify p, d and r as the adjoint variables associated with the x, y and w variables respectively. Using the sum rule we observe from (5.4.67) that

$$\begin{aligned}
&\partial^C\left(\langle q_i(t), f(t,x_i(t),u_i(t))\rangle + r_i|u(t)-u_i(t)|\right)\\
&\subset \partial^C\left[q_i(t).f(t,x_i(t),u_i(t))\right] + \partial^C\left[r_i|u(t)-u_i(t)|\right]\\
&= \{(a,0,0,b,0,0,c) : (a,b,c)\in \partial^C_{x,p,u}(q_i(t).f(t,x_i(t),u_i(t)))\}\\
&+\{(0,0,0,0,0,|u(t)-u_i(t)|, r_i\beta_i) : \beta_i\in\partial^C_u|u(t)-u_i(t)| \text{ and } \|\beta_i\|\le 1\}\\
&= \{(a,0,0,b,0,0,c+r_i\beta_i) : \|\beta_i\|\le 1 \text{ and } (a,b,c)\in\partial^C_{x,p,u}(q_i(t).f(t,x_i(t),u_i(t)))\}.
\end{aligned} \tag{5.4.69}$$

Thus, we get $\dot{d}_i = 0$ and $\dot{r}_i = 0$ and then $d_i(t) = d_i$ and $r_i(t) = r_i$, where d_i, r_i are constants. Hence we get

$$(-\dot{p}_i(t), \dot{x}_i(t), \xi_i(t)) \in \partial^C(q_i(t).f(t,x_i(t),u_i(t))) + (0,0,r_i\beta_i) \text{ with } \|\beta_i\|\le 1. \tag{5.4.70}$$

It follows from (5.4.68) that

$$\begin{aligned}
&(p_i(a), d_i, r_i, -q_i(b), -d_i, -r_i, \textstyle\int_{[a,b]}\mu_i(dt)) \in N^L_E(x_i(a), y_i(a)) \times \mathbb{R}^n \times \{(0,0,0,0)\}\\
&+\{(a,b,0,c,d,0,e) : (a,b,c,d,e)\in \partial^L\tilde{l}_i(x_i(a),y_i(a),x_i(b),y_i(b),\max\{h(t,x_i(t))\})\\
&\qquad +\varepsilon_i[\mathbb{B}\times\mathbb{B}\times\{0\}\times\{(0,0)\}\times\{1\}\times\{0\}].
\end{aligned} \tag{5.4.71}$$

This implies that

$$\begin{aligned}
&(p_i(a), d_i, -q_i(b), -d_i, \textstyle\int_{[a,b]}\mu_i(dt)) \in N^L_E(x_i(a), y_i(a)) \times \{(0,0,0)\}\\
&\qquad +\partial^L\tilde{l}_i(x_i(a),y_i(a),x_i(b),y_i(b),\max\{h(t,x_i(t))\})\\
&\qquad +\varepsilon_i(\mathbb{B}\times\mathbb{B})\times\{(0,0,0)\}
\end{aligned} \tag{5.4.72}$$

and

$$r_i = -\varepsilon_i.$$

From (5.4.72) we deduce that $\{\|\mu_i\|_{T.V}\}$, $\{d_i\}$ and $\{p_i(b)\}$ are bounded sequences. By (5.4.70) $\{p_i\}$ is uniformly bounded and $\{\dot{p}_i\}$ and $\{\xi_i\}$ are uniformly integrably bounded. We deduce that, following subsequence extraction,

$$p_i \to p \text{ uniformly}, \quad \xi_i \to \xi \text{ in the } L^1 \text{ norm}, \quad d_i \to d,$$

and

$$\mu_i \to \mu, \quad \gamma_i\mu_i(dt) \to \gamma\mu(dt) \text{ weakly}^*,$$

for some $p \in W^{1,1}$, $d \in \mathbb{R}^n$, $\xi \in L^1$, $\mu \in C^{\oplus}$ and some Borel measurable function γ, as $i \to \infty$. Furthermore,

$$\text{supp}\{\mu\} \subset \left\{ t : h\left(t, x^*(t)\right) = \max_{s \in [a,b]} h(s, x^*(s)) \right\}$$

and

$$\gamma(t) \in \bar{\partial}_x h(t, x^*(t)) \quad \mu\text{-a.e.}$$

where $q := p + \int \gamma \mu(ds)$.

By subsequence extraction we can have $\{\xi_i\}$ converging to ξ almost everywhere. A convergence analysis along the lines of the proof of Theorem 3.1 in [24] and an appeal to the upper semi continuity properties of limiting subdifferentials and normal cones allow us to pass to the limit in relationships (5.4.70) and (5.4.68) which gives the results

$$(-\dot{p}(t), \xi(t)) \in \partial^C \langle (q(t), f(t, x^*(t), u^*(t))) \rangle \text{ a.e. } t \in [a, b],$$

and

$$\xi(t) \in N^C_{U(t)}(u^*(t)) \ \text{ a.e. } \ t \in [a, b].$$

Now we will analyze the transversality condition (5.4.72). Recall that

$$\tilde{l}_i(x_i(a), y_i(a), x_i(b), y_i(b), \max_{s \in [a,b]} h(s, x(s))) > 0 \tag{5.4.73}$$

for all sufficiently large i. Otherwise a contradiction can be found.

Indeed, assume $\tilde{l}_i = 0$ for all sufficiently large i. Then we get

$$x_i(b) = y_i(b) = y_i(a), \ (x_i(a), x_i(b)) \in E, \ \max_{s \in [a,b]} h(s, x_i(s)) \leq 0, \ \|x_i - x^*\|_\infty \leq \varepsilon.$$

and

$$l(x_i(a), x_i(b)) \leq l(x^*(a), x^*(b)) - \varepsilon_i^2,$$

which is the violation of optimality of (x^*, u^*) (in the definition of $\tilde{l}_i$).

Set

$$z_i = \max_{s \in [a,b]} h(s, x_i(s))$$

and we get the following estimate for $\partial^L \tilde{l}_i$:

$$\begin{aligned} &\partial^L \tilde{l}_i(x_i(a), y_i(a), x_i(b), y_i(b), z_i) \subset \\ &\{(a, b, e, -e, c) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \exists\ \tilde{\lambda} \geq 0 \text{ such that } \tilde{\lambda} + |e| = 1 \text{ and} \\ &(a, b, c) \in \tilde{\lambda} \partial^L \max\{l(x, y) - l(x_i(a), y_i(a)) + \varepsilon_i^2, z\}|_{(x_i(a), y_i(a), z_i)}\}. \end{aligned} \tag{5.4.74}$$

See [67] and [25] for the proof.

Mimicking the proof of the Theorem 2.1 in [25] we obtained the required conclusions. We omit the details. ■

5.4.3 Proof of Theorem 5.3.3

We will prove now our final refinement proposed in Theorem 5.3.3. We recall that (x^*, u^*) is a $W^{1,1}$ local minimizer if (x^*, u^*) minimizes the cost over all admissible processes (x, u) such that for any $\varepsilon > 0$,

$$|x(t) - x^*(t)| \leq \varepsilon \quad \text{a.e.}$$

and

$$\|x - x^*\|_{W^{1,1}} = |x(a) - x^*(a)| + \int_a^b |\dot{x}(t) - \dot{x^*}(t)| dt \leq \varepsilon.$$

We consider the problem

$$(P') \quad \begin{cases} \text{Minimize } l(x(a), x(b)) \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \ \dot{y}(t) = |f(t, x(t), u(t)) - \dot{x^*}| \quad \text{a.e. } t \in [a, b] \\ \ h(t, x(t)) \leq 0 \quad \text{for all } \ t \in [a, b] \\ \ u(t) \in U(t) \quad \text{a.e. } \ t \in [a, b] \\ \ (x(a), x(b), y(a), y(b)) \in E \times \{0\} \times \mathbb{R}^n. \end{cases}$$

We need to show that if (x^*, u^*) is a $W^{1,1}$ local minimizer for (P), then $(x^*, y^* \equiv 0, u^*)$ is a strong local minimizer for (P'). Now (x^*, y^*, u^*) is an admissible process for (P'). Let us take any other admissible process (x, y, u) such that

$$|(x, y)(t) - (x^*, y^*)(t)| \leq \frac{\varepsilon}{2} \tag{5.4.75}$$

and suppose

$$l(x(a), x(b)) < l(x^*(a), x^*(b)).$$

By (5.4.75) we know that

$$|x(t) - x^*(t)| \leq \frac{\varepsilon}{2} \text{ for all } t$$

and

$$|y(t) - y^*(t)| \leq \frac{\varepsilon}{2} \text{ for all } t$$

But

$$y(t) = 0 + \int_a^b |f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))|dt.$$

So $|x(a) - x^*(a)| \leq \varepsilon$ and $\displaystyle\int_a^b |f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))|dt \leq \varepsilon$. It follows that (x, u) is admissible for (P) and

$$\|x - x^*\|_{W^{1,1}} = |x(a) - x^*(a)| + \int_a^b |\dot{x}(t) - \dot{x^*}(t)|dt \leq \varepsilon$$

and

$$l(x(a), x(b)) < l(x^*(a), x^*(b)).$$

So (x^*, u^*) is not a $W^{1,1}$ minimizer for (P) which is a contradiction.

Thus we conclude that if (x^*, u^*) is a $W^{1,1}$ local minimizer for (P), then $(x^*, y^* \equiv 0, u^*)$ is a strong local minimizer for (P').

Now we apply Theorem 5.3.2 to problem (P'). We can do that since problem (P') satisfies all the hypotheses - including **AH1** :

$$\begin{aligned} |\, |f(t, x, u) - \dot{x^*}| - |f(t, x', u') - \dot{x^*}|\, | &\leq |f(t, x, u) - \dot{x^*} - f(t, x', u') + \dot{x^*}| \\ &= |f(t, x, u) - f(t, x', u')| \\ &\leq k_x^f |x - x'| + k_u^f |u - u'|. \end{aligned} \tag{5.4.76}$$

Then there exist absolutely continuous functions $p \colon [a, b] \to \mathbb{R}^n$ and $\pi \colon [a, b] \to \mathbb{R}^n$, integrable functions $\xi : [a, b] \to \mathbb{R}^m$, and $\gamma : [a, b] \to \mathbb{R}^n$, a non-negative measure $\mu \in C^{\oplus}([a, b]; \mathbb{R})$, and a scalar $\lambda_0 \geq 0$ such that

$$\mu\{[a,b]\} + ||p,\pi||_\infty + \lambda_0 > 0, \tag{5.4.77}$$

$$(-\dot{p}(t), -\dot{\pi}(t), \xi(t)) \in \partial^C_{x,y,u}\langle (q,\rho), (f(t,x^*(t),u^*(t)), |f(t,x^*(t),u^*(t)) - \dot{x^*}|)\rangle$$
$$+\{0,0\} \times N^C_{U(t)}(u^*(t)) \quad \text{a.e.} \tag{5.4.78}$$

$$\forall \quad u \in U(t), \quad \langle q(t), f(t,x^*(t),u)\rangle \le \langle q(t), f(t,x^*(t),u^*(t))\rangle \quad \text{a.e.} \tag{5.4.79}$$

$$(q(a), -q(b)) \in N^L_E(x^*(a), x^*(b)) + \lambda_0 \partial^L l(x^*(a), x^*(b)), \tag{5.4.80}$$

$$(\rho(a), -\rho(b)) \in N^L_{\{0\}\times\mathbb{R}^n}(y^*(a), y^*(b)), \tag{5.4.81}$$

$$(\gamma_1(t), \gamma_2(t)) \in \partial^>_{x,y} h(t, x^*(t)) \quad \mu\text{-a.e.}, \tag{5.4.82}$$

$$\gamma_2(t) = 0, \tag{5.4.83}$$

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h\,(t, x^*(t)) = 0\}\,, \tag{5.4.84}$$

where

$$q(t) = \begin{cases} p(t) + \int_{[a,t)} \gamma_1(s)\mu(ds) & t \in [a,b) \\ p(t) + \int_{[a,b]} \gamma_1(s)\mu(ds) & t = b. \end{cases} \tag{5.4.85}$$

and

$$\rho(t) = \begin{cases} \pi(t) + \int_{[a,t)} \gamma_2(s)\mu(ds) & t \in [a,b) \\ \pi(t) + \int_{[a,b]} \gamma_2(s)\mu(ds) & t = b. \end{cases} \tag{5.4.86}$$

Now since by (5.4.82)–(5.4.83) $\gamma_2(t) = 0$, we have from (5.4.86) that $\rho(t) = \pi(t)$. Also from (5.4.78) we have $\dot{\pi}(t) = 0$ which implies that $\pi(t) = \text{constant}$. But from (5.4.81) and (5.4.86) we have

$$-\rho(b) = -\pi(b) \in N^L_{\mathbb{R}^n}(y^*(b)) = \{0\}.$$

So we obtain

$$\pi(t) \equiv 0.$$

Now using the Sum and Product rules for subdifferentials [67], we obtain from (5.4.78),

$$(-\dot{p}(t), \xi(t)) \in \partial^C_{x,u}\langle (q(t), f(t,x^*(t),u^*(t))\rangle + \{0\} \times N^C_{U(t)}(u^*(t)) \quad \text{a.e.} \tag{5.4.87}$$

Now surveying all the details and using $\pi(t) \equiv 0$ for all $t \in [a,b]$ we rewrite the necessary conditions (5.4.77)–(5.4.84) for the problem (P') in the following:

$$\mu\{[a,b]\} + ||p||_\infty + \lambda_0 > 0, \tag{5.4.88}$$

$$(-\dot{p}(t), \xi(t)) \in \partial^C_{x,u}\langle(q(t),(f(t,x^*(t),u^*(t))\rangle + \{0\} \times N^C_{U(t)}(u^*(t)) \quad \text{a.e.} \tag{5.4.89}$$

$$\forall \;\; u \in U(t), \quad \langle q(t), f(t,x^*(t),u)\rangle \le \langle q(t), f(t,x^*(t),u^*(t))\rangle \quad \text{a.e.} \tag{5.4.90}$$

$$(q(a), -q(b)) \in N^L_E(x^*(a),x^*(b)) + \lambda_0 \partial^L l(x^*(a),x^*(b)), \tag{5.4.91}$$

$$\gamma(t) \in \partial^>_x h(t,x^*(t)) \quad \mu\text{-a.e.}, \tag{5.4.92}$$

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h\,(t,x^*(t)) = 0\}\,, \tag{5.4.93}$$

where

$$q(t) = p(t) + \int_{[a,b]} \gamma(t)\mu(dt) \quad t \in [a,b]. \tag{5.4.94}$$

Finally, the Theorem 5.3.3 is proved and thus we prove that if (x^*, u^*) is a $W^{1,1}$ local minimizer for (P) even then the Theorem 5.3.2 holds true. ■

Chapter 6

Nonsmooth Maximum Principle for State and Mixed Constrained Problems

In this chapter we derive a new nonsmooth maximum principle for problem (P_m) with both pure state and mixed state-control constraints. Our approach is similar to what is done in the previous chapter.

6.1 Problem Statement and Assumptions

Consider the optimal control problem (P_m)

$$
(P_m) \quad \begin{cases} \text{Minimize } l(x(a), x(b)) + \displaystyle\int_a^b L(t, x(t), u(t))\ dt \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b], \\ \ h(t, x(t)) \leq 0 \quad \text{ for all } \ t \in [a, b], \\ \ g(t, x(t), u(t)) \leq 0 \quad \text{a.e. } t \in [a, b], \\ \ u(t) \in U \quad \text{ a.e. } \ t \in [a, b], \\ \ (x(a), x(b)) \in E. \end{cases}
$$

Here the interval is fixed. The functions l, f, L and h, and the sets U and E are as in Chapter 5. The function g describing the mixed constraints is defined in $[a, b] \times \mathbb{R}^n \times \mathbb{R}^k$

and takes values in $\mathbb{R}^m$, i.e., $g : [a,b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$. Problem (P_m) reduces to a *standard optimal control problem* whenever the state and mixed constraints are absent reducing to (S) in section 5.2. We note that Problem (P_m) differs from the Problem (P) in Chapter 5 because of the presence of *mixed constraints*. Here we drop the dependence of U on t to simply the analysis.

Again for (P_m) a pair (x, u) comprising an absolutely continuous function x (state or trajectory) and a measurable function u (control), is called an *admissible process* if it satisfies all the constraints. Throughout this chapter the pair (x^*, u^*) will always denote the solution of the optimal control problem under consideration.

We assume throughout this chapter the following basic assumptions:

B1 $(t, (x, u)) \to (L(t, (x, u)), f(t, (x, u)), g(t, (x, u))$ are $\mathcal{L} \times \mathcal{B}$-measurable,

B2 l is locally Lipschitz and the set E is closed.

Assumption **B1** differs from **BH1** since we add the $\mathcal{L} \times \mathcal{B}$-measurability of $(t, (x, u)) \to g(t, (x, u))$.

Before stating additional assumptions we set

$$S(t) := \{(x, u) \in \mathbb{R}^n \times U : \ g(t, x, u) \leq 0\} \tag{6.1.1}$$

and

$$S^*_\varepsilon(t) := \{(x, u) \in \mathbb{R}^n \times U : \ |x - x^*(t)| \leq \varepsilon, \ (x, u) \in S(t)\} .$$

Observe that we can rewrite the mixed constraint $g(t, x(t), u(t)) \leq 0$ as $(x(t), u(t)) \in S(t)$ where $S(t)$ is as in (6.1.1). We shall use such notation whenever it simplifies our analysis.

We also define the multifunction $S^u : [a,b] \times \mathbb{R}^n \to \mathbb{R}^k$ as

$$S^u(t, x) := \{u \in U : \ (x, u) \in S(t)\} ,$$

and $F^- : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ as

$$F^-(t, x) := \{(f(t, x, u), \ L(t, x, u)) : \ u \in S^u(t, x)\} . \tag{6.1.2}$$

For each $t \in [a,b]$, $S(t)$ is the graph of $x \to S^u(t, x)$ and thus we can say

$$(x, u) \in S(t) \Longrightarrow u \in S^u(t, x).$$

Our additional hypotheses are similar to those of Chapter 5. However, for completeness we state those here again. Recall that ϕ is a generic function defined in $[a,b] \times \mathbb{R}^n \times \mathbb{R}^k$ and taking values in $\mathbb{R}^n$ or $\mathbb{R}$. This function can be as f, g or L and in each case the Lipschitz parameters are then denoted by k_x^f, k_u^f and/or k_x^g, k_u^g and k_x^L, k_u^L respectively for [**H1**].

[**H1**] There exist constants k_x^ϕ and k_u^ϕ such that for every (x_i, u_i) $(i = 1, 2)$ and almost every $t \in [a,b]$ the following condition is satisfied: $|x_i - x^*(t)| \leq \varepsilon$, $u_i \in U$ we have

$$|\phi(t, x_1, u_1) - \phi(t, x_2, u_2)| \leq k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

[**H2**] The set $U \subset \mathbb{R}^k$ is compact.

[**H3**] There exists a constant $k_h > 0$ such that the function $x \to h(t,x)$ is Lipschitz of rank k_h for all $t \in [a,b]$. Furthermore for all x, the function $t \to h(t,x)$ is continuous except on a finite number of points in $]a,b[$ and at any point t_k the following holds:

$$\lim_{s \to t_k^-} h(s,x) \text{ exists}, \ \lim_{s \to t_k^-} h(s,x) \leq h(t_k, x), \text{ and } \lim_{s \to t_k^+} h(s,x) = h(t_k, x),$$

as in **AH3**.

[**BS**] There exists a constant M such that, for almost every t, all $(x,u) \in S_\varepsilon^*(t)$, $\eta \in N_U^L(u)$, $\gamma \in \mathbb{R}_+^m$, $\langle \gamma,\ g(t,x,u) \rangle = 0$:

$$(\alpha, \beta - \eta) \in \partial_{x,u}^L \langle \gamma,\ g(t,x,u) \rangle \implies |\gamma| \leq M |\beta|.$$

[**N**] For each $t \in [a,b]$ and $x \in \mathbb{R}^n$, there exists $u \in U(t)$ such that $g(t,x,u) \leq 0$.

[**C**] The set $F^-(t,x)$ is convex for all $(t,x) \in [a,b] \times \mathbb{R}^n$.

In our analysis, assumption [**BS**] plays a crucial role specially when we deal with mixed constrained problems. This is a *Mangasarian Fromowitz* type condition which is implied by other well known regularity assumptions on the mixed constraints under which Maximum Principles for mixed constrained problems were proved. These include linearly independence or positively linear independence of the gradients $\nabla_u g_i(t,x,u)$, for $i = 1, \ldots, m$ (see for example [30]). We emphasize mentioning that [**BS**] is the translation in our case of *bounded slope* condition for $S(t)$ (see [19] for a discussion) and it plays an important role in our setting.

Assumption [**C**] is a convexity hypothesis on the "velocity set". Observe that if the multifunction $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ defined as

$$F(t,x) := \{(f(t,x,u),\ L(t,x,u),\ g(t,x,u)) :\ u \in U\},$$

is convex, then [**C**] is automatically satisfied. Convexity of F can be easier to verify in some situations. As for [**N**] it simply guarantees that $S(t)$ is nonempty.

We recall that our assumptions have some consequences which are of importance in the forthcoming analysis. As in the previous chapter we can show the existence of integrable functions K_f and K_L such that

$$|f(t,x,u)| \leq K_f(t) \text{ a. e.}$$

and

$$|L(t,x,u)| \leq K_L(t) \text{ a. e.}$$

(see (5.1.1) and (5.1.2) in Chapter 5). Also

$$f(t,x,U),\ g(t,x,U) \text{ and } L(t,x,U) \ \text{ are compact for all } x \in x^*(t) + \varepsilon\mathbb{B}.$$

Finally note that here our control set U does not depend on t in contrast with Chapter 5. However, our results and proofs remain unchangeable when U depends on t but **AH2** of Chapter 5 is satisfied.

6.2 On Mixed Constraints

In this section, we intend to discuss some important properties of mixed constraints. Without loss of generality and to simplify our analysis we consider $L \equiv 0$ in the definition of the multifunction F^-. We also assume that f and g satisfy [**H1**] and that [**H2**], [**H3**], [**BS**], [**N**] and [**C**] hold.

Let $\varepsilon > 0$ and x^* be an absolutely continuous function such that

$$\dot{x}^*(t) \in F^-(t,x^*), \quad \text{a.e. } t \in [a,b]. \tag{6.2.3}$$

Define $X(t) = x^*(t) + \varepsilon\mathbb{B}$. Let $\mathcal{S}^*_{[a,b]}(E)$ be the set of all absolutely continuous functions x associated with a control $u : [a,b] \to U$ such that $x(t) \in X(t)$ for all $t \in [a,b]$, and satisfying the control system

$$\begin{cases} \dot{x}(t) &= f(t,x(t),u(t)) \text{ a.e.,} \\ 0 &\geq g(t,x(t),u(t)) \text{ a.e.,} \\ (x(a),x(b)) &\in E. \end{cases}$$

We say that an absolutely continuous function x is a feasible trajectory of F^- if $x(t) \in X(t)$ for all $t \in [a,b]$ and x satisfies (6.2.3). We denote the set of all F^--feasible trajectories associated with E to be

$$\mathcal{T}^*_{[a,b]}(E) := \left\{ x \in C([a,b];\mathbb{R}^n) : \ x \text{ is a } F^- \text{ trajectory and } (x(a),x(b)) \in E \right\}.$$

We now state some properties of F^-, S^u and S that will be of relevance in our analysis. In this regards, the following results in the form of lemma will be of help. We refer the reader to [43] for a complete report on such lemma.

Lemma 6.2.1 *Assume that our basic assumptions as well as* [**H1**] $-$ [**H3**], [**BS**], [**N**] *and* [**C**] *hold. Then*

a) the multifunctions S^u and F^- are non-empty and compact valued,

b) the multifunction F^- is $\mathcal{L} \times \mathcal{B}$ measurable,

c) for almost every $t \in [a,b]$ and all $x \in X(t)$ there exists an integrable function c such that for all $\gamma \in F^-(t,x)$ we have $|\gamma| \leq c(t)$,

d) there exist a $\bar{\epsilon} > 0$ and an integrable function K_F such that for almost every $t \in [a,b]$ and all $x, x' \in \{x : \ |x - x^*(t)| < \bar{\epsilon}\}$ we have

$$F^-(t,x) \subset F^-(t,x') + K_F(t)|x - x'|\mathbb{B},$$

e) $x \in \mathcal{S}^*_{[a,b]}(E)$ if and only if $x \in \mathcal{T}^*_{[a,b]}(E)$.

When the multifunction F^- fails to be convex, we consider well-known relaxation techniques: we turn to the convex hull of F^-, co $F^-(t,x)$. In this regard, it is worth to establish the relation between the differential inclusion $\dot{x}(t) \in \text{co } F^-(t,x(t))$ and the appropriate control system. In doing so, let $x \in W^{1,1}$ be such that $\dot{x}(t) \in \text{co } F^-(t,x(t))$ a.e.

Then, by Lemma 6.2.1 and Carathéodory's Theorem, there exist $(u_1(t), u_2(t), \dots, u_{n+1}(t))$ and $(\lambda_1(t), \dots, \lambda_{n+1}(t)) \in \Lambda$, where $u_i(t) \in U$ for $i = 1, \dots, n+1$ such that

$$(x, (u_1(t), u_2(t), \dots, u_{n+1}(t)), (\lambda_1(t), \dots, \lambda_{n+1}(t)))$$

is a solution to the system

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n+1} \lambda_i(t) f(t, x(t), u_i(t)), \quad \text{a.e.,} \\ g(t, x(t), u_i(t)) \leq 0, \ \ i = 1, \dots, n+1, \quad \text{a.e.,} \\ (\lambda_1(t), \dots, \lambda_{n+1}(t)) \in \Lambda, \quad \text{a.e.} \\ u_i(t) \in U, \quad \text{a.e. for } i = 1, \dots, n+1 \end{cases} \tag{6.2.4}$$

where

$$\Lambda := \{\lambda' \in \mathbb{R}^{n+1} : \lambda' \geq 0 \text{ and } \sum_{i=1}^{n+1} \lambda'_i = 1\}.$$

Observe that if f and g satisfy **[H1]** and **[BS]**, then the data of (6.2.4) satisfy analogous conditions.

To see this, let us define $v(t) = (u_1(t), u_2(t), \dots, u_{n+1}(t))$, $V \subset \mathbb{R}^{k \times (n+1)}$ with $V = U \times \dots U$ and

$$\tilde{f}(t, x, v, \lambda) = \sum_{i=1}^{n+1} \lambda_i f(t, x, u_i), \quad \tilde{g}(t, x, v, \lambda) = (g(t, x, u_1), \dots, g(t, x, u_{n+1})).$$

The control variable is now (v, λ). Then for almost every $t \in [a, b]$, for all $x, \ \ x \in X(t)$, $v, v' \in V$ and $\lambda, \lambda' \in \Lambda$ we have

$$\begin{aligned} |\tilde{f}(t, x, v, \lambda) - \tilde{f}(t, x', v', \lambda')| &= |\tilde{f}(t, x, v, \lambda) - \tilde{f}(t, x, v, \lambda') + \tilde{f}(t, x, v, \lambda') - \tilde{f}(t, x', v', \lambda')| \\ &\leq \sum_{i=1}^{n+1} \Big(|\lambda_i - \lambda'_i| \ |f(t, x, u_i)| + |f(t, x, u_i) - f(t, x', u'_i)| \Big) \\ &\leq \sqrt{n+1} K_f |\lambda - \lambda'| + k_x^f |x - x'| + \sqrt{n+1} k_u^f |v - v'|, \end{aligned}$$

where K_f is as defined in (5.1.1). So $\tilde{f}$ is Lipschitz with respect to (x, v).

Recall that for $z = (z_1, \dots, z_r)$ we have $\sum_{i=1}^{r} |z_i| \leq \sqrt{r} |z|$. Taking into account that $\tilde{g}$ does not depend on λ, it is easy to see that satisfies a condition similar to **[H1]** (see below). We now focus on **[BS]**. Set

$$\tilde{S}(t) = \{(x, v, \lambda) \in \mathbb{R}^n \times V \times \Lambda : \ \tilde{g}(t, x, v, \lambda) \leq 0\}.$$

Take $\gamma_i \in \mathbb{R}^m_+$ and set $\gamma = (\gamma_1, \ldots, \gamma_{n+1})$. For almost every $t \in [a, b]$, take any $x \in X(t)$, $(x, v, \lambda) \in \tilde{S}(t)$ and any $\gamma \in \mathbb{R}^{m\times(n+1)}_+$ such that $\langle \gamma, \tilde{g}(t, x, v, \lambda)\rangle = 0$. It follows that for each $i = 1, \ldots, n+1$ we have

$$\langle \gamma_i, g(t, x, u_i)\rangle = 0.$$

Take $(\eta, \xi) \in N^L_{V\times\Lambda}(v, \lambda)$. Then, by properties of the normal cones (see Proposition 6.41 in [60]), we have $\eta = (\eta_1, \ldots, \eta_{n+1})$, where $\eta_i \in N^L_U(u_i)$ and $\xi \in N^L_\Lambda(\lambda)$. Take $(\alpha, \beta - \eta, \chi - \xi) \in \partial^L_{x,v,\lambda}\langle \gamma, \tilde{g}(t, x, v, \lambda)\rangle$. Appealing to the sum rule of subdifferentials we have $(\alpha, \beta - \eta, \chi - \xi) \in \sum_{i=1}^{n+1} \partial^L_{x,v,\lambda}\langle \gamma_i, g(t, x, u_i)\rangle$. It is then an easy task to conclude that

$$\alpha = \sum_{i=1}^{n+1} \alpha_i, \quad \beta - \eta = (\beta_1 - \eta_1, \ldots, \beta_{n+1} - \eta_{n+1}), \quad \text{where } (\alpha_i, \beta_i - \eta_i) \in \partial^L_{x,u_i}\langle \gamma_i, g(t, x, u_i)\rangle.$$

By **[BS]** we have $|\gamma_i| \leq M|\beta_i|$ for each $i = 1, \ldots, n+1$. We conclude that

$$|\gamma| \leq \sum |\gamma_i| \leq M \sum |\beta_i| \leq \sqrt{n+1} M|\beta| \leq \sqrt{n+1} M|(\beta, \xi)|,$$

that is, $\tilde{g}$ satisfies a condition analogous to **[BS]** with parameter $M\sqrt{n+1}$.

For future reference we now state the following supporting hypotheses where $\tilde{f}$, $\tilde{g}$, V, v and Λ are as defined above:

[HS1] For almost every $t \in [a, b]$, for all $x \in X(t)$, $v, v' \in V$ and $\lambda, \lambda' \in \Lambda$ we have

$$|\tilde{f}(t, x, v, \lambda) - \tilde{f}(t, x', v', \lambda')| \leq \sqrt{n+1} K_f |\lambda - \lambda'| + k^f_x |x - x'| + \sqrt{n+1} k^f_u |v - v'|$$

and

$$|\tilde{g}(t, x, v, \lambda) - \tilde{g}(t, x', v', \lambda')| \leq (n+1) k^g_x |x - x'| + \sqrt{n+1} k^g_u |v - v'|.$$

[HS2] The set Λ are compact, V is closed and $\forall\ c > 0 : |v| \leq c, \ \forall\ v \in V$

[HS3] For almost every $t \in [a, b]$, all $x \in X(t)$ and all (v, λ) such that $(x, v, \lambda) \in \tilde{S}(t)$,

$(\eta, \xi) \in N^L_{V\times\Lambda}(v, \lambda)$ and any $\tilde{\gamma} \in \mathbb{R}^{m\times(n+1)}_+$ such that $\langle \tilde{\gamma}, \tilde{g}(t, x, v, \lambda)\rangle = 0$, we have

$$(\alpha, \beta - \eta, \chi - \xi) \in \partial^L_{x,v,\lambda}\langle \gamma, \tilde{g}(t, x, v, \lambda)\rangle \implies |\tilde{\gamma}| \leq \sqrt{n+1}M|(\beta, \xi)|.$$

6.3 Auxiliary Results

In this section we shall present some auxiliary results which are essential in developing our main results. For this purpose we turn to the problem (P_m) above when the state constraint is absent. We present Theorem 6.3.1 as an adaptation of Theorem 7.1 in [19] for (P_m) when the state constraint is absent.

Theorem 6.3.1 *Let (x^*, u^*) be a local $W^{1,1}$ minimizer for problem (P_m) in absence of* pure state constraints. *Assume that the basic assumptions as well as* **[H2]**, **[BS]** *and* **[HS1]**–**[HS3]** *hold and that f, g and L satisfy* **[H1]**. *Then there exist $p \in W^{1,1}([a, b]; \mathbb{R}^n)$, and a scalar $\lambda_0 \geq 0$ satisfying the* nontriviality condition*:*

$$||p||_\infty + \lambda_0 > 0, \tag{6.3.5}$$

the Euler adjoint inclusion*:*

$$\begin{aligned}(-\dot{p}(t), 0) \in \ & \partial^C_{x,u}\Big(\langle p(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t)) \\ & -K(t)|(p(t) + \lambda_0)|d_{S(t)}(x^*(t), u^*(t))\Big) \quad a.e.,\end{aligned} \tag{6.3.6}$$

the global Weierstrass condition*: for all $u \in S(t, x^*(t))$,*

$$\begin{aligned}&\langle p(t), f(t, x^*(t), u)\rangle - \lambda_0 L(t, x^*(t), u) \\ &\leq \langle p(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t)) \quad a.e.,\end{aligned} \tag{6.3.7}$$

and the transversality condition*:*

$$(p(a), -p(b)) \in N^L_E(x^*(a), x^*(b)) + \lambda_0\partial^L l(x^*(a), x^*(b)). \tag{6.3.8}$$

Above K is an integrable function defined in terms of the Lipschitz parameters and M in **[BS]**.

Remark 6.3.2 *Before proceeding it is important to note that in the statement of Theorem 7.1 in [19] the inclusion analogous to our (6.3.6) is written as*

$$(-\dot{p}(t), 0) \in \partial^C_{x,u}\Big(\langle p(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t))\Big) - N^C_{S(t)}(x^*(t), u^*(t)).$$

However what is proved in [19] is a sharper *version where* $N^C_{S(t)}(x^*(t), u^*(t))$ *is replaced by*

$$\partial^C_{x,u} K(t)|p(t) + \lambda_0| d_{S(t)}(x^*(t), u^*(t))$$

(see remark on pp. 4522 in [19]). If $L = 0$*, then* $|p(t) + \lambda_0|$ *is replaced by* $|p(t)|$*. Here and for reasons that will be clear later on, we use the above sharper version of the Euler adjoint inclusion.*

6.4 Nonsmooth Maximum Principle for (P_m)

We are now in position to state our main results for problem (P_m) in its full generality. As in the previous chapter we first present a new nonsmooth maximum principle in the vein of Theorem 6.3.1 when **[C]** holds. This is Proposition 6.4.1. Extension of this Proposition for the nonconvex case is presented as Theorem 6.4.2. Once more, the proofs of our results can be seen as based on [24] and [25] which in turn are based on [66] and [67].

We recall that in our study we consider the state constraints function $t \to h(t, x)$ as continuous except on a finite number of point in $]a, b[$ (see **[H3]**) instead of taking only upper semi-continuous.

6.4.1 The Convex Case

Our main results when convexity assumption is in force for the Problem (P_m) are stated in the following Proposition 6.4.1. The convexity restriction will be removed later on.

Proposition 6.4.1 Let (x^*, u^*) be a strong local minimizer for problem (P_m). Suppose that f, g and L satisfy **[H1]**, h satisfies **[H3]** and assumptions **[H2]**, **[BS]**, **[N]** and **[C]** along with the basic hypotheses hold. Then there exist $p \in W^{1,1}([a, b]; \mathbb{R}^n)$, $\gamma \in L^1([a, b]; \mathbb{R})$, a measure $\mu \in C^{\oplus}([a, b]; \mathbb{R})$, and a scalar $\lambda_0 \geq 0$ satisfying

(i) $\mu\{[a, b]\} + ||p||_\infty + \lambda_0 > 0$,

(ii) $(-\dot{p}(t), 0) \in \partial^C_{x,u}\big(\langle q(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t))\big) - N^C_{S(t)}(x^*(t), u^*(t))$ a.e.,

(iii) $\forall$ $(x^*(t), u) \in S(t)$,

$$\langle q(t), f(t, x^*(t), u)\rangle - \lambda_0 L(t, x^*(t), u) \leq \langle q(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t)) \text{ a.e.},$$

(iv) $(p(a), -q(b)) \in N_E^L(x^*(a), x^*(b)) + \lambda_0 \partial^L l(x^*(a), x^*(b))$,

(v) $\gamma(t) \in \bar{\partial} h(t, x^*(t))$ μ-a.e.,

(vi) $\text{supp}\{\mu\} \subset \{t \in [a,b] : h(t, x^*(t)) = 0\}$, where

$$q(t) = \begin{cases} p(t) + \int_{[a,t)} \gamma(s)\mu(ds) & t \in [a,b) \\ p(t) + \int_{[a,b]} \gamma(s)\mu(ds) & t = b. \end{cases} \tag{6.4.9}$$

where M is a constant as in **[BS]** *and the subdifferential $\bar{\partial}$ is as defined in* (5.3.6).

When $(x, u) \to g(t, x, u)$ satisfies some extra differentiable properties, the normal cone $N^C_{S(t)}$ can be expressed in terms of the derivatives of g in (ii). We refer the reader to [19] in this regard.

6.4.2 Nonconvex Case

We now state our main result for (P_m) covering the nonconvex case. As in Chapter 5 we replace the subdifferential $\bar{\partial}_x h$ by subdifferential $\partial_x^> h$ defined in (5.3.8).

Theorem 6.4.2 Let (x^*, u^*) be a strong local minimizer for problem (P_m). Assume that f, g and L satisfy **[H1]**, h satisfies **[H3]**. Assume also that **[H2]**, **[BS]**, **[N]** as well as the basic assumptions hold. Then there exist an absolutely continuous function p, an integrable function γ, a non-negative measure $\mu \in C^\oplus([a,b]; \mathbb{R})$ and a scalar $\lambda_0 \geq 0$ such that conditions (i)–(vi) of Proposition 6.4.1 hold with $\partial_x^> h$ as in (5.3.8) replacing $\bar{\partial}_x h$ and where q is as defined in (6.4.9).

We remark that the above Theorems keeps the significant feature of being a sufficient condition of optimality in the normal form for linear-convex problem. This follows directly from the observation that a straightforward adaptation of the proof of Proposition 4.1 in [24] proves our claim.

It is worth mentioning that the above theorem adapts easily when we assume (x^*, u^*) to be a weak local minimizer instead of a strong local minimizer. To see that it is sufficient to replace U by $U \cap \mathbb{B}_\varepsilon(u^*(t))$. Theorem 6.4.2 can also be extended to deal with a local

$W^{1,1}$-minimizer for (P_m). This can be accomplished as in the last steps of the proof of Lemma 9.4.1 in [67]. We refer readers to Theorem 5.3.3 (see also Lemma 9.4.1 in [67]) in Chapter 5 omitting the proof but for the sake of completeness we only state the result in following Theorem.

Theorem 6.4.3 Let (x^*, u^*) be merely a local $W^{1,1}$-minimizer for problem (P_m). Then the conclusions of Theorem 6.4.2 hold.

As one may expect that Theorems 6.4.2 and 6.4.3 subsume analogous Theorems 5.3.2 and 5.3.3 in Chapter 5. This can be viewed assuming the mixed constraint is absent. On the other hand, if pure state constraint is absent, then Theorem 6.4.2 reduces to Theorem 6.3.1.

Note that the proofs of the above results can be adapted to treat the case where $S(t)$ incorporates additional equality constraints. Also we point out mentioning the fact that more general results could possibly be obtained using our techniques but considering Theorem 2.1 instead of Theorem 7.1 in [19]. That would cover the case $(x, u) \in S(t)$ for a general set $S(t)$. However it is our belief that such cases should be best handled using differential inclusion techniques.

6.5 Proofs of the Main Results

Most of the steps of the proofs of our results consist in adaptation of techniques applied in Chapter 5 although we differ here and there in the approach.

Once again for simplicity we shall prove our results assuming $L \equiv 0$. The case of $L \neq 0$ is treated by a standard and well known technique.

6.5.1 Proof of Proposition 6.4.1

The validity of the Proposition is established for a a special case of (P_m) with separate endpoints conditions of the form $E = E_a \times \mathbb{R}^n$. We denote such problem by $(\mathcal{P})$.

$$
(\mathcal{P}) \quad \left\{
\begin{array}{l}
\text{Minimize } l(x(a), x(b)) \\
\text{subject to} \\
\begin{array}{rcllr}
\dot{x}(t) & = & f(t, x(t), u(t)) & \text{a.e.} & t \in [a, b] \\
u(t) & \in & U & \text{a.e.} & t \in [a, b] \\
h(t, x(t)) & \leq & 0 & \text{for all} & t \in [a, b] \\
g(t, x(t), u(t)) & \leq & 0 & \text{a.e.} & t \in [a, b] \\
(x(a), x(b)) & \in & \{x_a\} \times \mathbb{R}^n. & &
\end{array}
\end{array}
\right.
$$

Observe that $(\mathcal{P})$ differs from (Q) in Chapter 5 since the cost depends mostly on $x(a)$ and $x(b)$.

As in Chapter 5 the proof breaks into several steps.

Step 1: Penalize state constraint violation

We start by introducing a sequence of problems $(\mathcal{P}_i)$ associated with problem $(\mathcal{P})$.

We define a sequence of problems, called $(\mathcal{P}_i)$, where the state constraint is included into the cost by using the penalization technique as

$$
\left\{
\begin{array}{l}
\text{Minimize } l(x(a), x(b)) + i \displaystyle\int_a^b h^+(t, x(t))\, dt \\
\text{subject to} \\
\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b], \\
u(t) \in U \quad \text{a.e. } t \in [a, b], \\
g(t, x(t), u(t)) \leq 0 \quad \text{a.e. } t \in [a, b], \\
(x(a), x(b)) \in E_a \times \mathbb{R}^n.
\end{array}
\right.
$$

Here $h^+(t, x) := \max\{0, h(t, x)\}$.

In support of the penalization technique applied, we assume that the following interim hypothesis holds:

[IH2] $\lim\limits_{i \to \infty} \inf\{\mathcal{P}_i\} = \inf\{\mathcal{P}\}$.

Step 2: Application of Ekeland's Theorem

Set $\mathcal{W}$ to be the set of pairs (u,s), where $s \in \mathbb{R}^n$ and $u : [a,b] \to \mathbb{R}^k$ is a measurable function satisfying $u(t) \in U$ a.e. such that there exists a function x satisfying the differential equation $\dot{x}(t) = f(t,x(t),u(t))$ a.e., the mixed constraint $g(t,x(t),u(t)) \leq 0$ a.e., with $x(t) \in x^*(t) + \varepsilon\mathbb{B}$ for all $t \in [a,b]$, $x(a) \in E_a$ and $x(b) = s$. We provide $\mathcal{W}$ with the metric

$$\delta((u,s),(v,s')) := \int_a^b |u(t) - v(t)| \, dt + |s - s'|$$

and, for each integer i, set $C_i(u,s) := l(x(a),x(b)) + i \displaystyle\int_a^b h^+(t,x(t))\, dt$ where x is the trajectory corresponding to (u,s).

The space $(\mathcal{W},\delta)$ is a complete metric space in which the functional $C_i \colon \mathcal{W} \to \mathbb{R}$ is continuous. The δ is a $(\mathcal{W},\delta)$ metric. To show that $\mathcal{W}$ is a complete metric space take any Cauchy sequence (u_n,s_n) where $(u_n,s_n) \in \mathcal{W}$ for all n. Since $u_n \in L^1([a,b];\mathbb{R}^k)$ and $s \in \mathbb{R}^n$, we get $(u_n,s_n) \to (u,s)$ where u is integrable and $s \in \mathbb{R}^n$. Moreover there exists a subsequence (we do not relabel) such that $u_n(t) \to u(t)$ for almost every t. Let x_n be the trajectory associated with (u_n,s_n). Then, by Lemma 6.2.1 e), $x_n \in \mathcal{T}^*_{[a,b]}(E_a \times \mathbb{R}^n)$. Lemma 6.2.1 together with Theorem 2.5.3 in [67] allow us to deduce the existence of a subsequence such that $x_n \to x \in \mathcal{T}^*_{[a,b]}(E_a \times \mathbb{R}^n)$ uniformly and $\dot{x}_n \to \dot{x}$ weakly in L^1. Since $x \in \mathcal{T}^*_{[a,b]}(E_a \times \mathbb{R}^n)$ we can apply Lemma 6.2.1 d) to deduce the existence of a control $\tilde{u} : [a,b] \to U$ such that $g(t,x(t),\tilde{u}(t)) \leq 0$, $\dot{x}(t) = f(t,x(t),\tilde{u}(t))$ and $x(a) \in E_a$. Taking into account the continuity properties of f and g and the fact that for almost every $t \in [a,b]$

$$x_n(t) = x_n(a) + \int_a^t f(\sigma, x_n(\sigma), u_n(\sigma)) d\sigma,$$

$g(t,x_n(t),u_n(t)) \leq 0$, $x_n(t) \to x(t)$ and $u_n(t) \to u(t)$, we may further deduce that $\tilde{u}(t) = u(t)$ and $x(b) = s$. So $(u,s) \in \mathcal{W}$ proving that $\mathcal{W}$ is complete. The Lipschitz properties of l and h assert the continuity of C_i.

For each i we consider the optimization problem $(\mathcal{O}_i)$,

$$(\mathcal{O}_i) \quad \text{Min } \{C_i(u,s) : (u,s) \in \mathcal{W}\}.$$

For all integers i, $(u^*, x^*(b))$ is an admissible solution of $(\mathcal{O}_i)$ with

$$C_i(u^*, x^*(b)) = l(x^*(a), x^*(b)) = \inf \mathcal{P}.$$

Set $\varepsilon_i := C_i(u^*, x^*(b)) - \inf \mathcal{P}_i$. Since $C_i(u^*, x^*(b)) \geq \inf \mathcal{P}_i$ we have $\varepsilon_i \geq 0$. By **IH2** we deduce that $\lim_{i\to\infty} \varepsilon_i = 0$.

Ekeland's Theorem (see [67]) applies to each $(\mathcal{O}_i)$ and from its conclusions we conclude that for each i there exists a process (x_i, u_i) with

$$\int_a^b |u_i(t) - u^*(t)|\ dt + |s_i - x^*(b)| \leq \sqrt{\varepsilon_i},$$

solving the optimal control problem $(\mathcal{E}_i)$:

$$\text{Minimize} \quad l(x(a), x(b)) + i \int_a^b h^+(t, x(t))\, dt + \sqrt{\varepsilon_i} \int_a^b |u(t) - u_i(t)|\ \ dt + \sqrt{\varepsilon_i}|x(b) - x_i(b)|$$

subject to

$$\begin{array}{rcl} \dot{x}(t) & = & f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b], \\ g(t, x(t), u(t)) & \leq & 0 \quad \text{a.e. } t \in [a, b], \\ u(t) & \in & U \quad \text{a.e. } t \in [a, b], \\ x(t) & \in & x^*(t) + \varepsilon \mathbb{B}, \quad \text{for all } t \in [a, b] \\ x(a) & \in & E_a. \end{array}$$

The fact that $\varepsilon_i \to 0$ allows us to prove that $x_i(b) \to x^*(b)$ and u_i converges strongly to u^*. It follows that there exists a subsequence $\{u_i\}$ (we do not relabel) converging to u^* for almost every $t \in [a, b]$. Along the corresponding subsequence of problems $(\mathcal{E}_i)$ we deduce that x_i converges uniformly to x^*. Discarding initial terms of this sequence, if necessary, we guarantee that $x_i(t) \in x^*(t) + \frac{\varepsilon}{2}\mathbb{B}$ for almost every $t \in [a, b]$. Thus the sequence (x_i, u_i) solves a variant of $(\mathcal{E}_i)$ obtained when the state constraint is absent. From now on $(\mathcal{E}_i)$ denotes such sequence of problems.

Step 3: Study of optimality condition for the perturbed problem

Now each problem $(\mathcal{E}_i)$ satisfies the conditions under which Theorem 6.3.1 holds. Applying it we get the existence of an absolutely continuous function p_i and a scalar $\lambda_i \geq 0$ such that

$$(p_i(t), \lambda_i) \neq 0 \text{ for all } t, \tag{6.5.10}$$

$$\begin{aligned} (-\dot{p}_i(t), 0) \in \partial^C_{x,u} \langle p_i(t), f(t, x_i(t), u_i(t)) \rangle - i\lambda_i \sigma_i(t)(\gamma_i(t), 0) \\ -\sqrt{\varepsilon_i}\lambda_i(0, e_i(t)) - K(t)|p_i(t)|\partial^C_{x,u} d_{S(t)}(x_i(t), u_i(t)) \quad \text{a.e.} \end{aligned} \tag{6.5.11}$$

$$\begin{aligned} &(x_i(t), u) \in S(t) \Longrightarrow \\ &\langle p_i(t), f(t, x_i(t), u) \rangle - \sqrt{\varepsilon_i}\lambda_i\, |u - u_i(t)| \leq \langle p_i(t), f(t, x_i(t), u_i(t)) \rangle \text{ a.e.} \end{aligned} \tag{6.5.12}$$

$$(p_i(a), -p_i(b)) \in N^L_{E_a\times\mathbb{R}^n}(x_i(a), x_i(b)) + \lambda_i \partial^L l(x_i(a), x_i(b)) + \lambda_i\sqrt{\varepsilon_i}\ \{0\}\times\{\tau_i\} \quad (6.5.13)$$

for some $(\gamma_i(t), 0) \in \partial^C_{x,u} h(t, x_i(t))$, $\sigma_i(t) \in [0,1]$ with $\sigma_i(t) = 0$ if $h(t, x_i(t)) < h^+(t, x_i(t))$ and $|e_i(t)| \leq 1$ and $|\tau_i| \leq 1$.

As in Chapter 5 we define now the measures $\mu_i \in C^*([a,b];\mathbb{R})$ and $m_i \in C^*([a,b];\mathbb{R}^n)$ such that

$$\int_B d\mu_i = \int_B i\lambda_i\sigma_i(t)dt, \quad \text{and} \quad dm_i(t) = \dot{p}_i(t)dt.$$

Here B is any Borel set in $[a,b]$. Since $\sigma_i(t) = 0$ if $h(t, x_i(t)) < 0$, the measure μ_i has support in

$$\{t \in [a,b] :\ h(t, x_i(t)) = 0\}.$$

Set $\pi_i := p_i(a)$. For $t \in (a,b]$ we have

$$p_i(t) = \pi_i + \int_{[a,t)} dm_i(t). \quad (6.5.14)$$

Thus (6.5.13) can be rewritten as

$$\begin{aligned}\big(\pi_i, -\pi_i - \int_{[a,b]} dm_i(t)\big) &\in N^L_{E_a}(x_i(a)) \times \{0\} \\ &+ \lambda_i\partial^L l(x_i(a), x_i(b)) + \lambda_i\sqrt{\varepsilon_i}\{0\}\times\{\tau_i\}.\end{aligned} \quad (6.5.15)$$

Let us now turn to (6.5.11). Appealing to measurable selection theorems we can choose

$$(f_i^x(t), f_i^u(t)) \in \partial^C_{x,u} f(t, x_i(t), u_i(t)) \text{ a.e.} \quad \text{and} \quad (d_i^x(t), d_i^u(t)) \in \partial^C_{x,u} d_{S(t)}(x_i(t), u_i(t)) \text{ a.e.}$$

so that

$$-\int_B dm_i(t) = \int_B \Big[\langle p_i(t), f_i^x(t)\rangle - K(t)|p_i(t)|d_i^x(t)\Big]dt - \int_B \gamma_i(t)d\mu_i \quad (6.5.16)$$

$$0 = \int_B \Big[\langle p_i(t), f_i^u(t)\rangle - K(t)|p_i(t)|d_i^u(t)\Big]dt - \sqrt{\varepsilon_i}\lambda_i\int_B e_i(t)dt \quad (6.5.17)$$

for any Borel set $B \subset [a,b]$. Rescalling the multipliers, if needed, (6.5.10) leads to

$$|\pi_i| + |\mu_i| + \lambda_i = 1. \quad (6.5.18)$$

Define $P_i(t) := m_i([a,t[)$. Then P_i is an absolutely continuous function and, from (6.5.16) and (6.5.14), we get

$$-P_i(t) = \int_{[a,t[} \Big[\langle \pi_i + P_i(s), f_i^x(s)\rangle - K(s)|\pi_i + P_i(s)|\Big] d_i^x(s)\ ds \qquad (6.5.19)$$
$$- \int_{[a,t[} \gamma_i(t) d\mu_i(t).$$

By **[H3]**, we have $|\gamma_i(t)| \leq k_h$ a.e. Then

$$\Big|\int_{[a,t[} \gamma_i(t) d\mu_i(t)\Big| \leq \int_{[a,t[} k_h d\mu_i(t) = k_h \int_{[a,b]} d\mu_i(t) = k_h |\mu_i|.$$

By the properties of subdifferential and **[H1]** we deduce that for almost every $t \in [a,b]$, $|f_i^x(t)| \leq K_f$, $|d_i^x(t)| \leq 1$, where $K_f = \max\{k_f^x, k_f^u\}$. Then, from (6.5.19) we have

$$|P_i(t)| \leq v_i(t) + \int_a^t (K_f(s) + K(s))|P_i(s)|\ ds, \qquad (6.5.20)$$

where $v_i(t) := \Big|\int_{[a,t[} \langle \pi_i, f_i^x(s)\rangle\ ds + |\pi_i| \int_{[a,t[} K(s) d_i^x(s)\ ds + \int_{[a,t[} \gamma_i(s) d\mu_i(s)\Big|$.
It follows from (6.5.18) that

$$v_i(t) \leq |\pi_i|(\|K_f\|_1 + \|K\|_1) + k_h|\mu_i|. \qquad (6.5.21)$$

Thus v_i is integrable and we can apply Gronwall's Lemma (see Lemma 5.2.1) to (6.5.20) yielding

$$|P_i(t)| \leq v_i(t) + \int_a^t \exp\left(\int_a^s (K_f(\sigma) + K(\sigma)) d\sigma\right)(K_f(s) + K(s)) v_i(s)\ ds \quad \text{for all } t \in [a,b].$$

The inequality (6.5.21), **[H1]**, **[H3]**, Gronwall's Lemma (see Lemma 5.2.1), Lemma 6.2.1 and properties of subdifferential allow us to conclude that there exists a $K_1 > 0$ such that $|P_i(t)| \leq K_1$. Then $|m_i| \leq K_1$ and it follows from (6.5.14) and (6.5.18) that $|p_i(t)| \leq K_1 + 1$.

Step 4: Take limits

All the analysis above have been dealt with fixed $i \in \mathbb{N}$. Now we consider $i \to \infty$ and we take limits. In this respect recall that the sequence x_i converges uniformly to x^* and $u_i \to u^*$ almost everywhere. It is an easy task to conclude from (6.5.18) that, by subsequence extraction, if necessary, $\pi_i \to \pi$, $\lambda_i \to \lambda$ and $\mu_i \to \mu$ weakly* for some

$\pi \in \mathbb{R}^n$, $\lambda \geq 0$ and some measure μ. Moreover we have $|\mu_i| \to |\mu|$ and

$$|\pi| + |\lambda| + |\mu| = 1. \tag{6.5.22}$$

From $|m_i| \leq K_1$ we deduce the existence of a measure $m \in C^*([a,b];\mathbb{R}^n)$ such that $m_i \to m$ weakly*. Thus $|m_i| \to |m|$. Lemma 4.3 in [66] asserts the existence of a bounded variation function q such that $p_i(t) \to q(t)$ a.e. $t \in [a,b]$ where

$$q(t) := \pi + \int_{[a,t)} dm.$$

Moreover we have $\pi_i + \int_{[a,t)} dm_i \to \pi + \int_{[a,t)} dm$. In view of the properties of limiting subdifferentials and limiting normal cones, (6.5.15) and the above, we deduce that

$$(\pi, -\pi - \int_{[a,b]} dm(t)) \in N^L_{E_a}(x^*(a)) \times \{0\} + \lambda \partial^L l(x^*(a), x^*(b)).$$

The Lipschitz properties of the distance function and of f, the upper semi continuity of Clarke subdifferentials and Dunford-Pettis Theorem (see Theorem 2.5.2 in [67], for example) allow us to deduce that

$$(f_i^x, f_i^u, d_i^x, d_i^u, \gamma_i, e_i) \to (f^x, f^u, d^x, d^u, \gamma, e) \text{ weakly in } L^1$$

and, for almost every $t \in [a,b]$,

$$(f^x(t), f^u(t)) \in \partial^C_{x,u} f(t, x^*(t), u^*(t)), \quad (d^x(t), d^u(t)) \in \partial^C_{x,u} d_{S(t)}(x^*(t), u^*(t)),$$
$$\gamma(t) \in \partial^C_x h(t, x^*(t)), \quad |e(t)| \leq 1.$$

The properties of $\bar{\partial}_x h(t, x^*(t))$, the fact that $\partial^C_x h(t, x^*(t)) \subset \bar{\partial}_x h(t, x^*(t))$ and Lemma 4.5 in [66] allow us to deduce that $\gamma(t) \in \bar{\partial}_x h(t, x^*(t)) \quad \mu - a.e.$ Mimicking the approach in [24] we also conclude that

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h(t, x^*(t)) = 0\}.$$

Let us introduce now the absolutely continuous function p defined as

$$p(t) := q(t) + \int_{[a,t)} \gamma(s) d\mu(s).$$

Gathering together all our conclusions and taking (6.5.11)-(6.5.13) into account we deduce that (i)–(vi) of Proposition 6.4.1 hold.

Step 5: Show that [C] implies [IH2]

Our next step is to remove the interim hypothesis, that is, we show that **[C]** implies **[IH2]**.

This is done as in Chapter 5 with obvious changes. For each i choose a feasible process (x_i, u_i) for problem $(\mathcal{P}_i)$ such that $x_i(t) \in x^*(t) + \varepsilon\mathbb{B}$ and

$$l(x_i(a), x_i(b)) + i \int_a^b h^+(t, x_i(t))dt \leq \inf(\mathcal{P}_i) + \frac{1}{i}. \tag{6.5.23}$$

Recall the definition of F^- in (6.1.2). Taking into account Lemma 6.2.1, compactness arguments in the vein of Theorem 2.5.3 in [67] guarantee that $x_i \to x$ uniformly for some $x \in W^{1,1}$ such that $x(t) \in x^*(t) + \varepsilon\mathbb{B}$ and

$$\begin{cases} \dot{x}(t) & \in & F^-(t, x(t)) \\ (x(a), x(b)) & \in & E_a \times \mathbb{R}^n \end{cases}$$

By Lemma 6.2.1 we deduce the existence of a measurable function u such that

$$\begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)) \\ 0 & \geq & g(t, x(t), u(t)) \\ u & \in & U \\ (x(a), x(b)) & \in & E_a \times \mathbb{R}^n \end{cases}$$

The remaining of the proof goes as in Chapter 5. So we omit the details.

Step 6: Last step

We have derived necessary conditions for problem $(\mathcal{P})$, a special case of (P_m). We complete the proof with two more steps. First we establish the validity of Proposition 6.4.1 when $(x(a), x(b)) \in E_a \times E_b$. This can be done along the lines of [27]. Then we return to our original problem (P_m) with $(x(a), x(b)) \in E$. To show that Proposition 6.4.1 holds we consider an extra state y with $\dot{y}(t) = 0$, $(x(a), y(a)) \in E$ and $(x(a), x(b)) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x = y\}$ as in [25] (see also Chapter 5 for details), completing the proof. ■

6.5.2 Proof of Theorem 6.4.2

We again consider $L \equiv 0$ for simplicity.

It is important to note here that we apply Proposition 6.4.1 with the Euler adjoint Inclusion (EI) condition in (ii) replaced by the sharper version:

$(ii)'$ $(-\dot{p}(t), 0) \in$

$$\partial^C_{x,u}\big(\langle q(t), f(t, x^*(t), u^*(t))\rangle - \lambda_0 L(t, x^*(t), u^*(t)) - K(t)|q(t)| d^C_{S(t)}(x^*(t), u^*(t))\big) \text{ a.e.}$$

This is exactly what we obtain in the proof of Proposition 6.4.1 by assuming $L = 0$. If we consider $L \neq 0$, then $|q(t)|$ in (ii)' should be replaced by $|q(t) + \lambda_0|$. But in our analysis, as we are assuming $L = 0$ throughout, the (EI) in (ii)' is what we need.

The proof takes the following steps:

Step 1: *Auxiliary Result*

We again pay attention to the following 'minimax' optimal control problem where the state constraint functional $\max_{t\in[a,b]} h(t, x(t))$ appears in the cost. The problem $(\tilde{R}_m)$ differs from the problem $(\tilde{R})$ in Chapter 5 because of the presence of mixed constraints $g(t, x(t), u(t)) \leq 0$ a.e. in addition of state constraints.

$$(\tilde{R}_m) \begin{cases} \text{Minimize } \tilde{l}_m(x(a), x(b), \max_{t\in[a,b]} h(t, x(t))) \\ \text{over } x \in W^{1,1} \text{ and measurable functions } u \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [a, b] \\ g(t, x(t), u(t)) \leq 0 \text{ a.e. } t \in [a, b] \\ u(t) \in U \text{ a.e. } t \in [a, b] \\ (x(a), x(b)) \in E_a \times \mathbb{R}^n. \end{cases}$$

where $\tilde{l}_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $E_a \subset \mathbb{R}^n$ is a given closed set.

To deal with this problem, we need some auxiliary results which are essential to our development. Before stating the auxiliary results, we shall impose the following additional assumption on the cost functional $\tilde{l}_m$ to deal with 'minimax' type problems.

[H4] The function $\tilde{l}_m$ is Lipschitz continuous on a neighborhood of

$$(x^*(a), x^*(b), \max_{t\in[a,b]} h(t, x^*(t)))$$

and if $z' \geq z$, then we have $\tilde{l}_m(y, x, z') \geq \tilde{l}_m(y, x, z)$, for all $(y, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Now for the problem $(\tilde{R}_m)$ we have:

Proposition 6.5.1 Suppose that (x^*, u^*) is a strong local minimizer for problem $(\tilde{R}_m)$ and that **[H1]**– **[H4]** and **[N]** are satisfied. Then there exist an absolutely continuous function $p : [a, b] \to \mathbb{R}^n$, integrable function $\gamma : [a, b] \to \mathbb{R}^n$ and a non-negative measure $\mu \in C^{\oplus}([a, b]; \mathbb{R})$ such that

$$(-\dot{p}(t), 0) \in$$
$$\partial^C_{x,u}\big(\langle q(t), f(t, x^*(t), u^*(t))\rangle - K(t)|q(t)|d_{S(t)}(x^*(t), u^*(t))\big) \quad \text{a.e.} \tag{6.5.24}$$

$$(p(a), -q(b), \int_{[a,b)} \mu(ds)) \in$$
$$N^L_{E_a}(x^*(a)) \times \{0, 0\} + \partial^L \tilde{l}_m(x^*(a), x^*(b), \max_{t\in[a,b]} h(t, x^*(t)), \tag{6.5.25}$$

$$\gamma(t) \in \bar{\partial} h(t, x^*(t)) \quad \mu\text{-a.e.}, \tag{6.5.26}$$

$$\forall \ (x^*(t), u) \in S(t), \ \ \langle q(t), f(t, x^*(t), u)\rangle \le \langle q(t), f(t, x^*(t), u^*(t))\rangle \ \text{ a.e.}, \tag{6.5.27}$$

$$\text{supp}\{\mu\} \subset \left\{t \in [a, b] : h(t, x^*(t)) = \max_{s\in[a,b]} h(s, x^*(s))\right\}, \tag{6.5.28}$$

where q is defined as in (6.4.9).

Proof of Proposition 6.5.1:

Lemma 6.2.1 e) asserts that x^* minimizes

$$\tilde{l}_m(x(a), x(b), \max_{t\in[a,b]} h(t, x(t)))$$

over all the trajectories of the differential inclusion $\dot{x}(t) \in F^-(t, x(t))$ such that $x(a) \in E_a$ and $\|x - x^*\|_\infty < \varepsilon$. In view of **[H3]**, Theorem 2.7.3 in [67] asserts that x^* minimizes $\tilde{l}_m(x(a), x(b), \max_{t\in[a,b]} h(t, x(t)))$ over all $x \in W^{1,1}$ satisfying

$$\dot{x}(t) \in \text{co } F^-(t, x(t)), \quad x(a) \in E_a, \quad \|x - x^*\|_\infty < \varepsilon.$$

Finally Carathéodory's Theorem and Lemma 6.2.1 e) allows us to deduce that, for $y^* = \tilde{l}_m(x^*(a), x^*(b), z^*)$, $z^* = \max_{t\in[a,b]} h(t, x^*(t))$,

$$\{x^*, y^*, z^*, (u^*_1, \ldots, u^*_{n+1}) \equiv (u^*, \ldots, u^*), (\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_{n+1}) \equiv (1, 0, \ldots, 0)\}$$

is a strong minimizer for the optimal control problem

$$(O_m)\begin{cases}\text{Minimize } y(b)\\ \text{over } x\in W^{1,1} \ \text{ and measurable functions } u_1,\dots,u_{n+1},\lambda_1,\dots,\lambda_{n+1} \text{ satisfying}\\ \dot{x}(t)=\sum_{i=1}^{n+1}\lambda_i(t)f(t,x(t),u_i(t)),\quad \dot{y}(t)=0,\ \dot{z}(t)=0,\quad \text{a.e.},\\ g(t,x(t),u_i(t))\le 0,\ i=1,\dots,n+1,\quad \text{a.e.},\\ (\lambda_1(t),\dots,\lambda_{n+1}(t))\in\Lambda,\quad \text{a.e.},\\ u_i(t)\in U,\ i=1,\dots,n+1,\quad \text{a.e.},\\ h(t,x(t))-z(t)\le 0,\ \text{for all } t\in[a,b],\\ x(a),x(b),y(a),z(a))\in \operatorname{epi}\{\tilde{l}_m+\Psi_{E_a\times\mathbb{R}^n\times\mathbb{R}}\}.\end{cases}$$

Here $(\lambda_1,\dots,\lambda_{n+1})$, $(u_1,\dots,u_{n+1})$ are regarded as control variables.

This is a problem with convex velocity set. The applicability of Proposition 6.4.1 follows from [**HS1**]–[**HS3**] in section 6.2. After rewriting the conclusions of Proposition 6.4.1 we also get the required conditions with $\lambda_0=1$ (see Chapter 5 for details if needed). ■

Step 2: *Conclusions of Theorem 6.4.2*

In this concluding step, we prove the main result. Recall that we derive the necessary conditions with the more refined subdifferential $\partial_x^{>}h$ defined in (5.3.8) replacing $\bar{\partial}_x h$ in Proposition 6.4.1.

Also recall that (5.1.1) and (5.1.3) hold as well as a condition analogous to (5.1.2) for L. Now we consider the set $\mathcal{W}$ of all (x,u,e), where $x\in W^{1,1}$, u is a measurable function and $e\in\mathbb{R}^n$, satisfying $\|x-x^*\|_\infty\le\varepsilon\}$, $\dot{x}(t)=f(t,x(t),u(t))$, $(x(t),u(t))\in S(t)$ for almost every $t\in[a,b]$ and $(x(a),e)\in E$ (recall that $S(t)$ is defined in (6.1.1)). Define now the function $d_{\mathcal{W}}:\mathcal{W}\times\mathcal{W}\to\mathbb{R}$ by

$$d_{\mathcal{W}}((x,u,e),(x',u',e'))=|x(a)-x'(a)|+|e-e'|+\int_a^b|u(t)-u'(t)|dt \tag{6.5.29}$$

For all i, we choose $\varepsilon_i\downarrow 0$ and set

$$\tilde{l}_{mi}(x,y,x',y',z):=\max\{l(x,y)-l(x^*(a),x^*(b))+\varepsilon_i^2,z,|x'-y'|\}.$$

As we can verify $d_{\mathcal{W}}$ defines a metric on $\mathcal{W}$ and $(\mathcal{W},d_{\mathcal{W}})$ is a complete metric space and

$$(x,u,e)\to\tilde{l}_{mi}(x(a),e,x(b),e,\max_{t\in[a,b]}h(t,x(t)))$$

is continuous on $(\mathcal{W}, d_{\mathcal{W}})$. The set $\mathcal{W}$ is nonempty since $(x^*, u^*, x^*(b)) \in \mathcal{W}$. Moreover, we have

$$\tilde{l}_{mi}(x^*(a), x^*(b), x^*(b), x^*(b), \max_{t\in[a,b]} h(t, x^*(t))) = \varepsilon_i^2. \tag{6.5.30}$$

Taking into account Lemma 6.2.1 we can now apply compactness results in the vein of Theorem 2.5.3 in [67] (see [43] in this regard) deducing that

$$(x_i, u_i, e_i) \to (x, u, e) \in (\mathcal{W}, d_{\mathcal{W}}) \Longrightarrow \|x_i - x\|_\infty \to 0. \tag{6.5.31}$$

Next we consider the optimization problem

$$\text{Minimize } \{\tilde{l}_{mi}(x(a), e, x(b), e, \max_{t\in[a,b]} h(t, x(t))) : (x, u, e) \in \mathcal{W}\}.$$

Since $\tilde{l}_{mi}$ is non-negative, it follows from (6.5.30) that $(x^*, u^*, x^*(b))$ is an ε_i^2-minimizer for the above minimization problem. Ekeland's Theorem asserts the existence of a sequence $\{(x_i, u_i, e_i)\}$ in $\mathcal{W}$ such that, for each i, we have

$$\begin{gathered}\tilde{l}_{mi}(x_i(a), e_i, x_i(b), e_i, \max\nolimits_{t\in[a,b]} h(t, x_i(t))) \leq \\ \tilde{l}_{mi}(x(a), e, x(b), e, \max\nolimits_{t\in[a,b]} h(t, x(t))) + \varepsilon_i d_{\mathcal{W}}((x, u, e), (x_i, u_i, e_i))\end{gathered} \tag{6.5.32}$$

for all $(x, u, e) \in \mathcal{W}$ and

$$d_{\mathcal{W}}((x_i, u_i, e_i), (x^*, u^*, x^*(b))) \leq \varepsilon_i. \tag{6.5.33}$$

The inequality (6.5.33) implies that $e_i \to x^*(b)$ and $u_i \to u^*$ strongly in L^1. Then there exists a subsequence (we do not relabel) such that $u_i \to u^*$ almost everywhere and $x_i \to x^*$ uniformly. Let us define the arc $y_i \equiv e_i$. We have $y_i \to x^*(b)$ uniformly. By (6.5.32) we can now conclude that $(x_i, y_i, w_i \equiv 0, u_i)$ is a strong local minimizer for the optimal control problem with mixed constraints

$$(\tilde{R}_{mi}) \begin{cases} \text{Minimize } \tilde{l}_{mi}(x(a), y(a), x(b), y(b), \max_{t\in[a,b]} h(t, x(t))) \\ \quad + \varepsilon_i[|x(a) - x_i(a)| + |y(a) - y_i(a)| + w(b)] \\ \text{over } x, y, w \in W^{1,1} \text{ and measurable functions } u \text{ satisfying} \\ \quad \dot{x}(t) = f(t, x(t), u(t)), \dot{y}(t) = 0, \dot{w}(t) = |u(t) - u_i(t)| \text{ a.e.,} \\ \quad (x(t), y(t), w(t)), u(t)) \in \hat{S}(t) \text{ a.e.,} \\ \quad (x(a), y(a), w(a)) \in E \times \{0\}, \end{cases}$$

where

$$\hat{S}(t) = \{(x,y,w,u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times U : \; g(t,x,u) \le 0\}.$$

Observe that

$$(x,y,w,u) \in \hat{S}(t) \Longleftrightarrow (x,u) \in S(t), \tag{6.5.34}$$

where $S(t)$ is defined in (6.1.1). The data of $(\tilde{R}_{mi})$ satisfies all the assumptions of the Proposition 6.5.1. Applying it we deduce the existence of absolutely continuous functions $p_i^x, p_i^y, p_i^w \in W^{1,1}$, an integrable function γ_i and a non-negative measure $\mu_i \in C^{\oplus}([a,b];\mathbb{R})$ satisfying

$$\begin{aligned}
(a) \quad & (-\dot{p}_i^x(t), -\dot{p}_i^y(t), -\dot{p}_i^w(t), 0) \in \partial^C \big(\langle q_i^x(t), f(t,x_i(t),u_i(t))\rangle + p_i^w(t)|u_i(t)-u_i(t)| \\
& \qquad -K(t)|(q_i^x(t), p_i^y(t), p_i^w(t))| d_{\hat{S}(t)}(x_i(t), y_i(t), 0, u_i(t))\big) \text{ a.e.}, \\
(b) \quad & \forall \;\; (x_i(t),u) \in S(t), \\
& \langle q_i(t), f(t,x_i(t),u)\rangle + p_i^w(t)|u-u_i(t)|) \le \langle q_i(t), f(t,x_i(t),u_i(t))\rangle \;\; \text{a.e.} \;, \\
(c) \quad & (p_i^x(a), p_i^y(a), p_i^w(a), -q_i(b), -p_i^y(b), -p_i^w(b), \textstyle\int_{[a,b]} \mu_i(dt)) \\
& \in N^L_{E\times 0}(x_i(a), y_i(a), w_i(a)) \times \{0,0,0,0\} \\
& +\partial^L\{\tilde{l}_{mi}(x_i(a), y_i(a), x_i(b), y_i(b), \max_{t\in[a,b]} h(t,x_i(t))) \\
& \qquad +\varepsilon_i[|x_i(a)-x_i(a)| + |y_i(a)-y_i(a)| + w_i(b)]\}, \\
(d) \quad & \gamma_i(t) \in \bar{\partial}_x h(t,x_i(t)) \quad \mu\text{-a.e.}, \\
(e) \quad & \operatorname{supp}\{\mu_i\} \subset \big\{t : h(t,x_i(t)) = \max_{s\in[a,b]} h(s,x_i(s))\big\}.
\end{aligned}$$

where $q_i(t) := p_i^x(t) + \int_{[a,t)} \gamma_i(s)\mu_i(ds)$ if $t < b$, $q_i(b) := p_i^x(b) + \int_{[a,b]} \gamma_i(s)\mu_i(ds)$ in the above relations.

Now using the Sum Rule in [12] and taking into account the fact that $\dot{p}_i^y = 0$ and $\dot{p}_i^w = 0$ imply $p_i^y(t) = p_i^y$ and $p_i^w(t) = p_i^w$ and (6.5.34), we get from (a) above

$$\begin{aligned}
(-\dot{p}_i(t), 0) \in \partial^C \langle q_i(t), f(t,x_i(t),u_i(t))\rangle + (0, p_i^w \beta_i(t)) \\
-K(t)|(q_i^x(t), p_i^y(t), p_i^w(t))| \partial^C d_{S(t)}(x_i(t), u_i(t))
\end{aligned} \tag{6.5.35}$$

with $\|\beta_i(t)\| \le 1$. Also from the condition (c), we get

$$\begin{aligned}
(p_i^x(a), p_i^y, -q_i(b), -p_i^y, \textstyle\int_{[a,b]} \mu_i(dt)) \in N^L_E(x_i(a), y_i(a)) \times \{(0,0,0)\} \\
+\partial^L \tilde{l}_{mi}(x_i(a), y_i(a), x_i(b), y_i(b), \max\{h(t,x_i(t))\}) + \varepsilon_i(\mathbb{B}\times\mathbb{B}) \times \{(0,0,0)\}
\end{aligned} \tag{6.5.36}$$

and $q_i^w = -\varepsilon_i$.

From (6.5.36), we conclude that $\{\|\mu_i\|_{T.V}\}$, $\{p_i^y\}$ and $\{p_i(b)\}$ are all bounded sequences. Then from (6.5.35) we can deduce that $\{p_i\}$ is uniformly bounded and $\{\dot{p}_i\}$ is uniformly integrably bounded. We deduce that, following subsequence extraction,

$$p_i \to p \text{ uniformly}, \quad p_i^y \to p^y,$$

and

$$\mu_i \to \mu, \qquad \gamma_i \mu_i(dt) \to \gamma\mu(dt) \text{ weakly}^*,$$

for some $p \in W^{1,1}$, $p^y \in \mathbb{R}^n$, $\mu \in C^{\oplus}$ and some Borel measurable function γ. It is then a simple matter to see that supp$\{\mu\}$ is a subset of $\big\{t : h(t, x^*(t)) = \max_{s\in[a,b]} h(s, x^*(s))\big\}$ and that $\gamma(t) \in \bar{\partial}_x h(t, x^*(t))$ μ-a.e.

We now introduce $q := p + \int \gamma\mu(ds)$. A convergence analysis along the lines of the proof of Theorem 5.3.2 in Chapter 5 and an appeal to the upper semi continuity properties of limiting subdifferentials and normal cones allow us to pass to the limit in relationships (6.5.35) leading to

$$\begin{gathered}(-\dot{p}(t), 0) \in \\ \partial^C\big(\langle q(t), f(t, x^*(t), u^*(t))\rangle - K(t)|(q(t), p^y)| d_{S(t)}(x^*(t), u^*(t))\big) \text{ a.e. } t \in [a, b].\end{gathered} \tag{6.5.37}$$

From (b) we also deduce that

$$\forall \ (x^*(t), u) \in S(t), \quad \langle q(t), f(t, x^*(t), u)\rangle \le \langle q(t), f(t, x^*(t), u^*(t))\rangle \text{ a.e.}$$

Now from (6.5.36) we get

$$\tilde{l}_{mi}(x_i(a), y_i(a), x_i(b), y_i(b), \max_{s\in[a,b]} h(s, x(s))) > 0. \tag{6.5.38}$$

for all sufficiently large i. See details in [67]. Set $z_i = \max_{s\in[a,b]} h(s, x_i(s))$.

Following the proof of Proposition 9.5.5 in [67] we get

$$\begin{gathered}\partial^L \tilde{l}_{mi}(x_i(a), y_i(a), x_i(b), y_i(b), z_i) \subset \\ \{(a, b, e, -e, c) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \exists\ \tilde{\lambda} \ge 0, \quad \tilde{\lambda} + |e| = 1 \\ \text{and } (a, b, c) \in \tilde{\lambda}\partial^L \max\{l(x, y) - l(x_i(a), y_i(a)) + \varepsilon_i^2, z\}|_{(x_i(a), y_i(a), z_i)}\}.\end{gathered} \tag{6.5.39}$$

Thus the results from the limiting subdifferential of $\tilde{l}_m$ together with (6.5.36) allow us to conclude that

$$\begin{cases} p_i^y = -q_i(b) \\ \tilde{\lambda}_i + |q_i(b)| = 1 \\ \|\mu_i\|_{T.V.} = c = \tilde{\lambda}_i(1-\alpha_i) \\ (p_i(a), -q_i(b)) \in N_E^L(x_i(a), y_i(a)) + \alpha_i \tilde{\lambda}_i \partial^L l(x_i(a), y_i(a)) + \varepsilon_i(b_1, b_2) \end{cases}$$

Also, we have $\mu_i = 0$ if $z_i \leq 0$, by (6.5.36), since $z_i \leq 0$ implies

$$\tilde{l}_{mi}(x, y, x', y', z) := \max\{l(x, y) - l(x^*(a), x^*(b)) + \varepsilon_i^2, |x' - y'|\}$$

for (x, y, x', y', z) near $(x_i(a), y_i(a), x_i(b), y_i(b), z_i)$ which in turns implies $\alpha_i = 1$ and thus consequently $\|\mu_i\|_{T.V.} = 0$.

Set $\lambda_i = \alpha_i \tilde{\lambda}_i$. Then $\|\mu_i\|_{T.V.} = \tilde{\lambda}_i(1-\alpha_i) = 1 - |q_i(b)| - \lambda_i$, and this gives

$$\lambda_i + \|\mu_i\|_{T.V.} + |q_i(b)| = 1. \tag{6.5.40}$$

Now taking the limit as $i \to \infty$ we get, along a subsequence, $\lambda_i \to \lambda$, for some $\lambda \geq 0$ and consequently

$$(p(a), -q(b)) \in \lambda \partial^L l(x^*(a), x^*(b)) + N_E^L(x^*(a), x^*(b))$$

and

$$\lambda + \|\mu\|_{T.V.} + |q(b)| = 1.$$

Finally we observe that p^y in (6.5.37) is actually $q(b)$. Combining all the above results we see that the Theorem 6.4.2 is almost proved except the replacement of $\bar{\partial}_x h$ by $\partial_x^> h$. This is done at the last part of the proof of Theorem 5.3.2 in Chapter 5 along the lines of the proof of Proposition 9.5.5 in [67]. This completes the proof. ∎

Chapter 7

Conclusions

In this concluding Chapter, we present a summary of the main *Contributions* of our work. We also propose *future work* for more research on state constrained optimal control problems and their applications in mathematical biology/epidemiology as well as biomedical engineering where state and/or mixed constraints may have influential roles. We also pose some open issues which require further extensive research and investigations in this challenging area.

7.1 Contributions

Our main contributions have been presented mainly in Chapters 3, 5 and 6. The other chapters reported in this thesis act as associated materials in developing the results of the contributed chapters. Our contributions include both theory and application of optimal control problems with constraints. In the theoretical part, we have derived *two sets* of nonsmooth necessary conditions of optimality for constrained optimal control problems in the form of maximum principles with appropriate assumptions. In the application part, we have developed new control strategy of SEIR epidemic model by vaccination. In order to motivate the readers to the importance of optimal control theory to practice, we have presented some references on real problems of optimal control in the beginning of Chapter 3.

Also in Chapter 3, a new and so far more realistic control strategy of a SEIR epidemic model for human infectious diseases has been proposed where the earlier model proposed by Neilan and Lenhart (2010) in [54] have been modified by replacing the so-called *isoperi-*

metric constraints by more general *mixed constraints*. Our numerical results support the theoretical data with normal form of minimizers. We have also studied some other situations of our model, such as, state constrained case and both state and mixed constraints case numerically. But we opt to keep such study out of this thesis since it is not yet complete (no analytical study).

In Chapter 5 a new nonsmooth maximum principles for optimal control problems with state constraints have been derived. First we proved our results when the velocity set is *convex*, but later on this restriction was removed to cover a larger class of *nonconvex* problems. This chapter comprises the convex case in Proposition 5.3.1 and the nonconvex case in Theorem 5.3.2 along with their proofs in section 5.4.

In Chapter 6 a new nonsmmoth maximum principle for problem with both pure state constraints and mixed state-control constraints have been derived. This chapter includes our results of the convex case in Proposition 6.4.1 and the nonconvex case in Theorem 6.4.2 along with their proofs in section 6.5. It is worth mentioning that our results in Chapter 6 subsumes those in Chapter 5. In fact one can get the results in Chapter 5 applying the results of Chapter 6 when no mixed constraints are present. Thus Chapter 6 is a generalization of Chapter 5.

7.2 Future Works

Optimal Control is a vast area of distinct research in the Dynamic Optimization, but our work has touched a little part of it. Further research is required to investigate further our work as well. Specially the application of new theorem to real problems in presence of constraints remains an open issue.

In this regard, it is worth mentioning our discussion on implementing the optimal control techniques to SEIR epidemic model in Chapter 3 by imposing state constraints in the existing data set for obtaining the new solution analytically. We recall that we have only discussed the state constrained model numerically. It is obvious that the analytical solution of such model is quite a challenging work because of the nonlinearity of the dynamic equations in one hand and the presence of measure due to state constraints in other hand. Although the numerical simulation gives quite a motivating result, the analytical validation of the results as well as the cross-checking with multipliers are still desirable.

The another future work in hand involves the mixed constraints considered in Chapter

6 is also needed. Here we considered the mixed constraints set in the form $g(t, x, u) \leq 0$, a.e. $t \in [a, b]$. A usual question in this regard is that what happens when the set is in the form of $(x(t), u(t)) \in S(t)$, a.e. $t \in [a, b]$? An appropriate answer to this question is, in our believe, simple but needs to be carefully studied. In fact, we believe that the main ideas to treat such problem do not differ greatly from our approach.

Research on Optimal Control with State Constraints is still an open issue. There are many questions still unsolved and/or not properly exploited, specially for the presence of measure (due to state constraints) in the problems. We believe that extensive and continuous involvement in optimal control research may result in to answer many questions and also may bring tremendous achievements for the generation to come.

Appendix A

Appendix

In this Appendix, we present some essential tools in the form of definitions, lemmas, propositions and theorems which are adapted from different literature. Since those are available in existing literature, we do not present any proof in most cases. We use those in our analysis as helping tools as some of those have significant roles in our developments.

A.1 On Functions

Definition A.1.1 (Equi-continuous Functions) *Let X be a metric space and let us define a family of functions $\mathcal{F} := \{f_\alpha : X \to \mathbb{R} : \ \alpha \in A, \text{ for any index set } A\}$. Then $\mathcal{F}$ is called* equi-continuous *if*

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 : |f(x) - f(y)| < \varepsilon \ \ \forall f \in \mathcal{F}, \ \forall x, y \in X : \|x - y\| \leq \delta.$$

Definition A.1.2 (Pointwise Bounded Functions) *The family of functions $\mathcal{F}$ is called* pointwise bounded functions *if*

$$\forall x \in X \ \exists \ M_x > 0 : |f(x)| \leq M_x \ \ \forall f \in \mathcal{F}.$$

Definition A.1.3 (Uniformly Bounded Functions) *The family of functions $\mathcal{F}$ is called* uniformly bounded functions *if*

$$\exists \ M > 0 : |f(x)| \leq M \ \ \forall f \in \mathcal{F}, \ \forall x \in X.$$

A.2 Banach-Alaoglu's Theorem

Definition A.2.1 (Weak Convergence) *Let X be a normed linear vector space and X^* be the dual space of X. A sequence $\{x_n\}$ is said to* converge weakly *to $x \in X$ if for all $x^* \in X^*$ we have $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$. In this case we write $x_n \to x$ weakly.*

Definition A.2.2 (Weak* Convergence) *Let X be a normed linear vector space and X^* be the dual space of X. A sequence $\{x_n^*\}$ in X^* is said to* converge weak-star *or weak* to the element $x^* \in X^*$ if $\langle x, x_n^* \rangle \to \langle x, x^* \rangle$ for all $x \in X$. In this case we write $x_n^* \to x^*$ weak*.*

Theorem A.2.3 (Banach-Alaoglu's Theorem [48]) *If X is a normed vector space, then the closed unit ball of X^* is compact in the weak* topology.*

Theorem A.2.4 ***(Sequential Banach-Alaoglu's Theorem [48])*** *If X is a separable normed vector space, then the closed unit ball of X^* is sequentially compact in the weak* topology.*

Recall that if

$$\{f_n\}, \quad f_n \in L^\infty([a,b];\mathbb{R}) : \|f_n\|_\infty \leq K,$$

then by Alouglu's Theorem we have

$$f_n(t) \to f(t) \;\; \forall\, t \in [a,b] \;\; \text{for some } f \in L^\infty.$$

A.3 Measure Theory

([37],[62]) Length, area and volume, as well as probability, are the instances of the measure concept. A measure is a set function, that is, an assignment of a number $\mu(A)$ to each set A in a certain class.

Definition A.3.1 (Field or Algebra) *Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a field (or algebra) iff $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union, that is,*

(a) $\quad \Omega \in \mathcal{F}$

(b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c is the complement of A relative to Ω.

(c) If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$

Definition A.3.2 (σ-Field (or σ-Algebra)) *Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a σ-field (or σ-algebra) in Ω if $\mathcal{F}$ has the following properties:*

(i) $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c is the complement of A relative to Ω.

(iii) If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition A.3.3 (Measurable space) *If $\mathcal{F}$ is a σ-algebra in Ω.Then Ω is called a measurable space and the members of $\mathcal{F}$ are called the measurable set in Ω.*

Definition A.3.4 (Measurable mapping) *If X is a measurable space, and Y is a topological space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X, for every open set V in Y.*

Definition A.3.5 (Measure) *Let $E = [a, b]$ be a given interval and let $\mathcal{M} \subset E$ be a $\sigma -$ algebra. A set function $\mu : \mathcal{M} \to [0, \infty]$ on the $\sigma -$ algebra $\mathcal{M}$ is called a measure if the following properties hold.*

- Semi-Positive-Definite: $0 \leq \mu(A) \leq (b - a)$ *for all* $A \in \mathcal{M}$.
- Triviality: $\mu(\emptyset) = 0$.
- Monotonicity: $\mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{M}$, $A \subset B$.
- Countable Additivity: *if* $A = \cup_{n=1}^{\infty} A_n$, *then* $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$

  for pairwise disjoint collections $\{A_n\} \in \mathcal{M}$.

A.4 Convexity of Sets and Functions

Definition A.4.1 (Convex Set) *Let $S \subset \mathbb{R}^n$ be a set on a vector space. Then S is said to be convex if $\forall s_1, s_2 \in S$ and any $\alpha \in [0, 1]$ we have,*

$$\alpha s_1 + (1 - \alpha) s_2 \in S.$$

That is, a line segment joining any pair of points s_1 and s_2 belonging to S also entirely belongs to S.

Geometrical configuration is shown in Figure A.1.

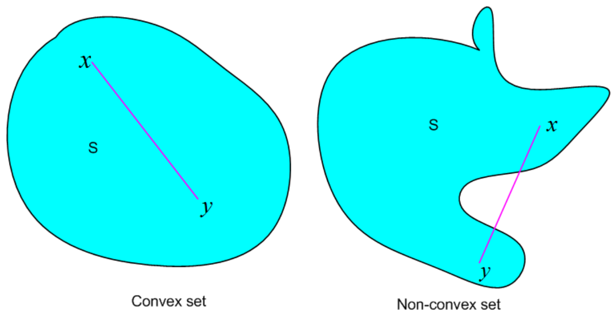

Figure A.1: Convex and Non-convex sets.

Properties of convex set

- $\emptyset$ set and *singleton set* are convex sets.
- The union of two convex sets is not necessarily convex, but the intersection of convex sets is convex.

Definition A.4.2 (Convex Functions) *Let $C \subset \mathbb{R}^n$ and consider a function $f : C \to \mathbb{R}^*$. Then f is called convex, if for all $u, v \in C$ and for any $\alpha \in [0, 1]$, the following holds:*

$$f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha)f(v).$$

A geometrical configuration of convex function is shown in Figure A.2.

A.5 On Multifunctions

Definition A.5.1 (Measurable Multifunction) *A multifunction $\Gamma : \Omega \to \mathbb{R}^n$ is said to be* measurable*, if the set,*

$$\{\omega \in \Omega : \Gamma(\omega) \cap C \neq \varnothing\}$$

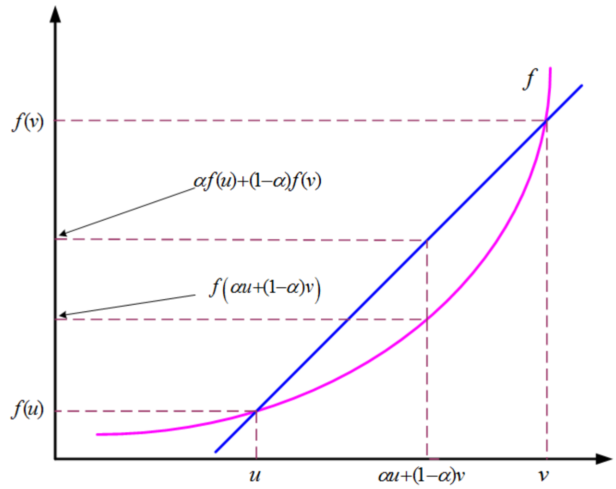

Figure A.2: Convex functions.

is Lebesgue measurable (or $\mathcal{L}$-measurable) for all open set $C \subset \mathbb{R}^n$ and the set $\Gamma(\omega) \subset \mathbb{R}^n$ is an element of the σ-algebra of Lebesgue measurable set.

Now let $\Gamma : [a, b] \times \mathbb{R}^m \to \mathbb{R}^n$ be a multifunction. Then Γ is called $\mathcal{L} \times \mathcal{B}$ measurable *if the set*

$$\{(t, x) \in [a, b] \times \mathbb{R}^m : \Gamma(t, x) \cap C \neq \emptyset\}$$

in $[a, b] \times \mathbb{R}^m$ belongs to the product σ-algebra $\mathcal{L} \times \mathcal{B}$ of the subset of $[a, b] \times \mathbb{R}^m$.

Definition A.5.2 (Measurable Selection) *Let $\Gamma : \Omega \to \mathbb{R}^n$ be a multifunction. We call a function $\gamma : \Omega \to \mathbb{R}^n$ to be a* measurable selection *for Γ, if*

(i) γ *is Lebesgue measurable,*

(ii) $\gamma(t) \in \Gamma(t)$ *a.e.*

Theorem A.5.3 (Measurable Selection Theorem (Theorem 2.3.11 in [67])) *Let $\Gamma : [a, b] \to \mathbb{R}^n$ be a nonempty multifunction. Assume that Γ is closed and measurable on $[a, b]$. Then there exists a measurable function (also called measurable selection) $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(t) \in \Gamma(t)$ for all $t \in [a, b]$.*

Theorem A.5.4 (Aumann's Measurable Selection Theorem [67]) *Let $\Gamma : [a, b] \to \mathbb{R}^n$ be a nonempty multifunction. Assume that Gr Γ is $\mathcal{L} \times \mathcal{B}^n$ measurable. Then there exists a measurable function (also called measurable selection) $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(t) \in \Gamma(t)$ for all $t \in [a, b]$.*

Theorem A.5.5 (Relaxation Theorem) *(Theorem 2.7.2, [67]) Let* $\Omega \subset [a,b] \times \mathbb{R}^n$ *be a relatively open set and let the multifunction* $F : \Omega \to \mathbb{R}^n$ *be a* $\mathcal{L} \times \mathcal{B}^n$ *measurable, closed and nonempty. Assume also that there exist integrable functions* $k \in L^1$ *and* $c \in L^1$ *such that*

$$F(t,x') \subset F(t,x'') + k(t)\mathbb{B} \ \ \textit{for all} \ \ (t,x'),(t,x'') \in \Omega$$

and

$$F(t,x) \subset c(t)\mathbb{B} \ \ \textit{for all} \ \ (t,x) \in \Omega.$$

Let us take any relaxed F*-trajectory* x *with* $Gr\ x \subset \Omega$ *and any* $\epsilon > 0$*. Then there exists an ordinary* F*-trajectory* y *that satisfies* $y(a) = x(a)$ *and*

$$\max_{t\in[a,b]} |y(t) - x(t)| \leq \varepsilon.$$

A.6 Useful Results and Lemmas

Lemma A.6.1 (Gronwall's Inequality in Differential Form) *(Lemma 2.4.4 in [67]) Let* $p : [a,b] \to \mathbb{R}^n$ *be an absolutely continuous function. Assume that there exist non-negative intregrable function* k *and* v *such that*

$$|\frac{d}{dt}p(t)| \leq k(t)|p(t)| + v(t) \quad a.e.t \in [a,b].$$

Then

$$|p(t)| \leq \exp\Big(\int_a^t k(\sigma)d\sigma\Big)\Big[|p(a)| + \int_a^t \exp\Big(-\int_a^\tau k(\sigma)d\sigma\Big)v(\tau)d\tau\Big] \quad \forall t \in [a,b].$$

Lemma A.6.2 (Gronwall's Inequality in Integral Form [69]) *Let* x *a be real continuous function and* K *and* v *be nonnegative integrable functions defined in* $[a,b]$ *. We suppose that on* $[a,b]$ *the following inequality holds:*

$$|x(t)| \leq v(t) + \int_a^t K(\tau)|x(\tau)| \ d\tau. \tag{1.6.1}$$

Then

$$|x(t)| \leq v(t) + \int_a^t \exp\big(\int_s^t K(\sigma)\ d\sigma\big)K(s)v(s)\ ds.$$

This is a well known result but not so easy to find in the literature in this general form. So we add the proof for completeness.

Proof of Lemma A.6.2 (as in [69]): Consider the absolutely continuous function

$$y(t) := \int_a^t K(s)|x(s)|\ ds.$$

For almost every $t \in [a, b]$ we have $\dot{y}(t) = K(t)|x(t)|$. By (1.6.1) we deduce that

$$\begin{aligned} \dot{y}(t) &\leq K(t)v(t) + K(t)\int_a^t K(t)|x(t)|\ dt \\ &= K(t)v(t) + K(t)y(t). \end{aligned}$$

Set $l(t) := \exp\Big(\int_a^t -K(\sigma)\ d\sigma\Big)$ ($l(t) > 0$). Multiply the previous inequality by $l(t)$. Then

$$l(t)\dot{y}(t) - l(t)K(t)y(t) \leq l(t)K(t)v(t). \tag{1.6.2}$$

But $l(t)\dot{y}(t) - K(t)l(t)y(t) = \frac{d}{dt}\big[l(t)y(t)\big]$. Integrating (1.6.2) and taking into account that $l(a)y(a) = 0$ we get, for all $t \in [a, b]$,

$$l(t)y(t) \leq \int_a^t l(s)K(s)v(s)\ ds.$$

Dividing this inequality by $l(t)$ we have

$$\begin{aligned} y(t) &\leq \exp\Big(\int_a^t K(\tau)d\tau\Big)\int_a^t \exp\Big(-\int_a^s K(\sigma)\ d\sigma\Big)K(s)v(s)\ ds \\ &= \int_a^t \exp\Big(\int_s^t K(\sigma)\ d\sigma\Big)K(s)v(s)\ ds. \end{aligned}$$

Now from (1.6.1) we have $|x(t)| \leq y(t) + v(t)$. So

$$|x(t)| \leq v(t) + \int_a^t \exp\Big(\int_s^t K(\sigma)\ d\sigma\Big)K(s)v(s)\ ds$$

as required. ∎

Theorem A.6.3 (Ekeland's Theorem [67]) *Let V be a complete metric space with associated metric Δ, a lower semicontinuous function $F : V \to \mathbb{R} \cup \{+\infty\}$ which is bounded from below and a point $u \in dom\ F$.*

If u is an $\varepsilon-$ minimizer (or almost minimizer) for F, that is,

$$F(u) \leq \mathit{inf}\ F + \varepsilon, \mathit{for\ some}\ \varepsilon > 0,$$

then for every $\lambda > 0$ there exists a nearby point $v \in \mathit{dom}\ F$ *satisfying* $\Delta(u, v) \leq \lambda$ *which is an actual minimizer for a slightly perturbed function, such that*

(i) $F(v) \leq F(u)$,

(ii) $F(v) \leq F(w) + \left(\frac{\varepsilon}{\lambda}\right) \Delta(w, v)$, *for all* $w \neq v$ *in* V.

A geometrical configuration of Ekeland's variational principle is shown in Figure A.3.

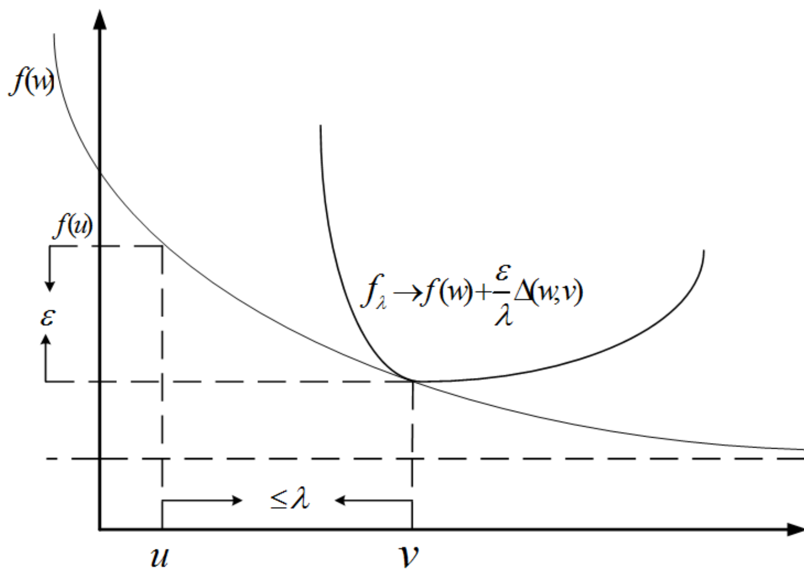

Figure A.3: Graphical representation of Ekeland's variational principle (source: [23, 67]).

Definition A.6.4 *A function $g : [a, b] \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a* Carathéodory's function, *if g satisfies the following conditions:*

(i) $g(., u)$ *is Lebesgue measurable for all* $u \in \mathbb{R}^m$,

(ii) $g(t, .)$ *is continuous for each* $t \in [a, b]$.

Theorem A.6.5 (Carathéodory's Theorem) *(Theorem 17.1 [59]) Let S be any set of points and directions (points at infinity) in $\mathbb{R}^n$ and let $C = co\ S$ (convex hull of S).*

Then $x \in C$ if and only if x can be expressed as a convex combination of no more than $n+1$ of the points and directions in S (not necessarily distinct).

Theorem A.6.6 (Arzela-Ascoli's Theorem ([62], page 245)) *Take a pointwise equicontinuous family $\mathcal{F}$ of function $f : X \to \mathbb{R}$ where X is a separable space.Then every sequence $\{f_n\}$, $\quad f_n \in \mathcal{F}$ has a subsequence that converges uniformly on every compact subsets of X.*

Recall that $\{f_n\}$ converges uniformly if

$$\exists\ f : X \to \mathbb{R} : \forall\ \varepsilon > 0\ \exists\ n \in \mathbb{N} : \|f_n - f\|_\infty = \max_{x \in X} |f_n(x) - f(x)| < \varepsilon.$$

Lemma A.6.7 (Duntord-Petti's Criterion) *Take a $\mathcal{F} \subset L^1$. Any sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$ has a subsequence which is weakly convergent in L^1 iff*

$$\forall\ \varepsilon > 0\ \ \exists\ \delta > 0 : \int_B |u(x)| dx \leq \varepsilon\ \ \forall\ u \in \mathcal{F}\ \ , \forall\ B \textit{ measurable set with measure } \ \leq \delta.$$

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